Tracking Performance under Time Delay and Asynchronicity in Distributed Camera Systems

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Abstract—Distributed camera systems are typically used to simultaneously track multiple targets. Communication between cameras enables the ability to monitor targets in a complex environment where occlusion happens. Asynchronicity and time delay are among the most important factors that affect tracking error. We provide a feedback law controlling the pan and tilt of each camera to track the centroid of a cluster of point targets in the field of view. Assuming that the estimates for target motion are available only infrequently and asynchronously when occlusion happens, we compute a worst-case lower bound on the frequency for exchanging estimates between cameras. Our results may help camera system designers to determine the response time and tracking ability for distributed camera systems.

I. INTRODUCTION

Recent advancements in technology have dramatically reduced the cost of manufacturing and installing distributed camera systems. Research in image processing and computer vision has enabled using cameras to track moving targets with high accuracy in real time. Distributed camera systems have the advantage of cooperatively tracking targets even when occlusion happens, i.e., when the field of view (FOV) of some cameras is blocked by obstacles. By communication, cameras with occlusion may obtain estimates of the blocked targets from other cameras monitoring the targets from different viewing angles. Communication delays and asynchronicity exist in most such distributed systems [1]. Controller design under time-delay has been a long lasting focus in the control literature c.f. [2]–[5]. The problem of asynchronicity has gained much recent interest due to the research thrusts in cooperative control and sensor networks [6]–[8].

There have been ongoing efforts to generalize Lyapunov stability theory to systems with hybrid nature [9]–[13]. In this paper, we develop methods based on input to state stability (ISS) for discrete-time systems [14]–[17] and apply them to a simplified perspective dynamic model [18]–[20] of the distributed camera systems to establish relationship between tracking error, delay, and asynchronicity. By explicitly determining an ISS Lyapunov function for the camerat racking system, one can estimate the size of a neighborhood to which the state will converge in equilibrium. This gives a measure of the system performance for different possible values of communication/computation delays and different control gains. Such a metric is highly desirable in applications involving control/scheduler co-design; if a certain level of tracking performance is required, we can schedule the state estimation/computation of the control law so that the control interval never exceeds a deadline, which we choose together with control gain and inter-measuremnt interval in such a way that the performance requirements are met.

In section II, we briefly review the perspective dynamics that model the motion of a point target in the image plane of a camera. We then introduce tracking controllers for a single camera to track one or a cluster of point targets in section III. In section IV, we consider the effect of time delay and asynchronicity on the tracking performance for two cameras tracking multiple targets. Simulation results will be presented in section V.

II. THE PERSPECTIVE DYNAMICAL SYSTEM

We consider cameras with controllable pan and tilt. We first introduce a model that describes the motion of a point target in the image plane of a camera.

Consider a single camera observing the space \( \mathbb{R}^3 \). We may establish the camera coordinate frame using three orthogonal unit vectors \( i, j, k \) defining the three axes. It is required that the origin is at the lens of the camera, and \( i \) and \( j \) span a plane that is parallel to the image plane of the camera. Hence \( k \) is perpendicular to the image plane. Rotation around \( i \) is the tilt motion and rotation around \( j \) is the pan motion. Since a camera is often fixed in location. We assume that the origin of the inertial frame coincides with the origin of the camera frame. When the pan and tilt motion are performed, the body coordinate frame rotates with the camera. The orientation of the camera in the inertial frame can be described by a \( 3 \times 3 \) matrix \( g \) that belongs to the special orthogonal group \( SO(3) \) i.e. \( gg^T = I \).

Let \( r_f \) represent a point in \( \mathbb{R}^3 \) (see fig. 1). The position of the point in the camera’s coordinate frame is \( r = g^{-1}r_f \). Taking time derivatives on both sides of \( r_f = gr \), and
We also define \( \dot{g} = g\dot{\Omega} \) where \( \dot{\Omega} \) is a 3 \times 3 skew symmetric matrix, we obtain \( \dot{r}_f = g(\Omega r + \dot{r}) \). The \( \dot{\Omega} \) here is also a matrix representation of the angular velocity vector \( \dot{\Omega} \) of the camera. This angular velocity can be written as \( \dot{\Omega} = u\dot{v} + v\dot{j} + 0k \) where \( u \) and \( v \) stand for the pan and tilt controls.

If we define a velocity vector \( \dot{b} = g^{-1}\dot{r}_f \), then we have \( \dot{r} = -\dot{\Omega} \times r + \dot{b} \). Now consider the projection of the vector \( \dot{r} \) onto the image plane of the camera with distance equal to unit length to the lens. We can write \( \dot{r} = r_1\dot{i} + r_2\dot{j} + r_3\dot{k} \) and \( \dot{b} = b_1\dot{i} + b_2\dot{j} + b_3\dot{k} \). Let the projection of \( \dot{r} \) be described as \( \dot{x} = x_1\dot{i} + x_2\dot{j} \). One has the following relationships between \( (r_1, r_2, r_3) \) and \( (x_1, x_2) \):

\[
x_1 = \frac{r_1}{r_3} \quad \text{and} \quad x_2 = \frac{r_2}{r_3}.
\]

(1)

The differential equations that \((x_1, x_2)\) satisfy are called the perspective dynamics. We now derive the perspective dynamical system assuming that \( b_1, b_2, \) and \( b_3 \) are constant. First, \( \dot{r} = -\dot{\Omega} \times r + \dot{b} \) implies that

\[
\begin{align*}
\dot{r}_1 &= b_1 - r_3v \\
\dot{r}_2 &= b_2 + r_3u \\
\dot{r}_3 &= b_3 - r_2u + r_1v.
\end{align*}
\]

(2)

We also define \((B_1, B_2, B_3)\) as

\[
B_1 = \frac{b_1}{r_3}, \quad B_2 = \frac{b_2}{r_3}, \quad \text{and} \quad B_3 = \frac{b_3}{r_3}.
\]

(3)

Taking the time derivatives of \( x_1 = \frac{r_1}{r_3} \) we have,

\[
\dot{x}_1 = -\frac{r_1\dot{r}_3}{r_3^2} + \frac{\dot{r}_1}{r_3} = x_1x_2u - (x_1^2 + 1)v + B_1 - B_3x_1.
\]

(4)

Taking the time derivatives of \( x_2 = \frac{r_2}{r_3} \) yields

\[
\dot{x}_2 = -\frac{r_2\dot{r}_3}{r_3^2} + \frac{\dot{r}_2}{r_3} = (x_2^2 + 1)u - x_1x_2v + B_2 - B_3x_2.
\]

(5)

Here, \( x_1, x_2, B_1, \) and \( B_2 \) are the position and velocity of the projected point in the image plane. They are directly measured by the camera. But \( B_3 \) can not be directly observed.

In the target tracking problem, we want to control the value of \( x_1 \) and \( x_2 \) so that the target is always in the field of view. If the camera is far away from the target, a reasonable assumption to make in this case is that \( B_3 \) is sufficiently small to be neglected. Since \( B_1 \) and \( B_2 \) can be directly observed, we can study the subsystem system formed by \((x_1, x_2)\) as

\[
\begin{align*}
\dot{x}_1 &= x_1x_2u - (x_1^2 + 1)v + B_1 \\
\dot{x}_2 &= (x_2^2 + 1)u - x_1x_2v + B_2.
\end{align*}
\]

(6)

In the following sections, we design controllers for the simplified perspective dynamics (6).

### III. Controller Design for Single Camera

In this section, we study the problem of controlling a single camera to track multiple targets in continuous time. When there is only one target to track, we design a controller for a single camera system to keep the target at the center of the FOV. When there are multiple targets, the controller keeps the centroids of the targets at the center of the FOV.

#### A. Tracking multiple targets with single camera

Suppose there are \( N \) targets, where \( N \geq 1 \). Define the centroid of all targets as \((x_{c1}, x_{c2})\) where \( x_{c1} = \frac{1}{N} \sum_{i=1}^{N} x_{i1} \) and \( x_{c2} = \frac{1}{N} \sum_{i=1}^{N} x_{i2} \), and \( i \) is the index for the targets. Then the equations for \( \dot{x}_{c1} \) and \( \dot{x}_{c2} \) are

\[
\begin{align*}
\dot{x}_{c1} &= \frac{1}{N} \left[ \sum_{i=1}^{N} x_{i1}x_{i2}u - \frac{1}{N} \sum_{i=1}^{N} (x_{i1}^2 + 1)v + \frac{1}{N} \sum_{i=1}^{N} B_{i1} \right] \\
\dot{x}_{c2} &= \frac{1}{N} \left[ \sum_{i=1}^{N} (x_{i2}^2 + 1)u - \frac{1}{N} \sum_{i=1}^{N} x_{i1}x_{i2}v + \frac{1}{N} \sum_{i=1}^{N} B_{i2} \right].
\end{align*}
\]

(7)

We now define the matrix

\[
G_c = \frac{1}{N} \left[ \sum_{i=1}^{N} x_{i1}x_{i2} - \frac{1}{N} \sum_{i=1}^{N} (x_{i1}^2 + 1) - \frac{1}{N} \sum_{i=1}^{N} x_{i1}x_{i2} \right].
\]

(8)

One can verify that \( \text{det}(G_c) > 0 \) by the induction method. Therefore, the control law can be obtained by

\[
\begin{bmatrix} u \\ v \end{bmatrix} = G_c^{-1} A \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} - \frac{1}{N} \left[ \sum_{i=1}^{N} B_{i1} \right] - \frac{1}{N} \left[ \sum_{i=1}^{N} B_{i2} \right].
\]

(9)

where \( A \) is a 2 \times 2 Hurwitz matrix. In the idealized case that all the states are known exactly, the equilibrium state (which corresponds to the centroid being at the center of the FOV) is globally asymptotically stable.

Occlusion happens when one or more targets are blocked by obstacles in the FOV of one camera. In this case, communication between cameras is necessary for tracking the targets. For example, every camera in the distributed camera system developed in [1] has the ability to request estimates about the blocked targets from another camera that has clear view of the targets. The estimates obtained from other cameras suffer time-delay and asynchronicity. We investigate how these factors affect the tracking error.

A typical pan and tilt camera has embedded computers that compute the control actions. Therefore, control of the cameras happen discretely. We discretize the perspective dynamical system (6) under the assumption that the control for pan and tilt happens accurately at time instants \( t_k \) where \( k = 0, 1, 2, \ldots \) and \( \Delta t = t_k - t_{k-1} \) is constant for all \( k \). Let \( G_k = G(t_k) \) and define

\[
F_k = \begin{bmatrix} B_1(t_k) \\ B_2(t_k) \end{bmatrix}, \quad \mathbf{u}_k = \begin{bmatrix} u(t_k) \\ v(t_k) \end{bmatrix}.
\]

(10)

Then the discrete perspective dynamics for one camera are

\[
x(t_{k+1}) = x(t_k) + G_k \mathbf{u}_k \Delta t + F_k \Delta t.
\]

(11)
B. Tracking a single occluded target with delay

Consider the following setup. Camera C1 is tracking a single target; the goal is to maintain the target as close as possible to the center of the FOV. The pan/tilt control \( u \) for the camera is calculated with an interval of \( T \) seconds, and simultaneously with the release of each control command, the camera accepts a measurement of the image-plane position \( x \) and velocity \( F \) of the target. Suppose now that the target is occluded in the FOV of C1. Camera C1 then requests estimates of the state of the occluded target from other cameras in the distributed system. Due to communication delay and necessity of converting the estimates from the measuring camera’s reference frame to the reference frame of C1, the state measurements \( x(t_k) \) and \( F(t_k) \) at C1 are available with some delay. Owing to the finite time to compute the control law, the corresponding pan/tilt control is available at time \( t_{k+m} = t_k + mT \), where \( m \) is a positive integer, \( m > 1 \). In practical situations, both \( m \) and \( T \) may vary; we assume here that both are constant and known, and represent the worst-case delay.

If we could measure the state and compute the corresponding control effort without computation delay, the discretized control law for tracking a single target at time \( t_j \) would be

\[
u_j = G_j^{-1}(Ax(t_j) - F_j)
\]

Due to computation delay and communication delays when the target becomes occluded, however, our control effort calculations at time \( t_j \) must rely on estimates of \( x(t_j) \) based on measurements made at some previous time, \( t_k \), \( k < j \leq k + m \). A simple estimator can be constructed to obtain the estimates recursively as follows:

\[
\hat{x}(t_{j+1}) = x(t_j) + G_ju_jT + F_jT \quad (13)
\]

\[
\hat{F}_{j+1} = \hat{F}_j \quad (14)
\]

\[
u_j = \hat{G}_j^{-1}(A\hat{x}(t_j) - \hat{F}_j) \quad (15)
\]

where \( x(t_k) \) and \( F_k \) are the measured values of the target’s position and velocity, respectively, and are assumed to be known exactly. The hatted values represent estimates, based on the previous available measurement. If at time \( t_j \) a measurement of the state is not available, the \( \hat{x} \) and \( \hat{F} \) are used instead of the exact values, so that \( x = \hat{x} \) and \( F = \hat{F} \) in the above equations.

If the estimates in (13)-(15) are used, then the value of \( \hat{x} \) at time \( t_{k+1} \) will be (assuming that a measurement is made at time \( t_k \)):

\[
\hat{x}(t_{k+1}) = x(t_k) + G_k(\hat{G}_k^{-1}(A\hat{x}(t_k) - \hat{F}_k))T + F_kT \quad (16)
\]

We write \( \hat{x}(t_k) \) as \( \hat{x}(t_k) = x(t_k) + \xi_k \), where \( \xi_k \) is the state estimation error. We can also define a velocity estimation error, \( \xi^F_k \), where \( \hat{F}_k = F_k + \xi^F_k \). These errors usually grow with the length of the inter-measurement time \( mT \). In this paper, we assume that the errors in estimates of the target’s position and velocity, \( \xi_k \) and \( \xi^F_k \), respectively, are independent random variables, and furthermore that \( \xi_k \) is uniformly distributed over the set \( \mathcal{X} = \{ \phi : \| \phi \| \leq L(\Delta t)^q \} \), while \( \xi^F_k \) is uniformly distributed over \( \mathcal{X} = \{ \psi : \| \psi \| \leq L(\Delta t)^r \} \). Here \( L \in \mathbb{R}^+ \) and \( q \in \mathbb{Z}^+ \) are known, and \( \Delta t \) is the time since the last available measurement.

We pick \( A \) to be a diagonal matrix, \( A = -K_kI \), where \( I \) is the \( 2 \times 2 \) identity matrix, and \( K_k \) is a positive scalar. This means that \( K_k \) may be interpreted as an adjustable control gain. Minimizing \( \| \hat{x}(t_{k+1}) \| \) with respect to \( K_k \) in eq. (13), we see that we should set \( K_k = \frac{1}{|I|} \). In fact, if we assumed that estimation errors are negligible, this control gain would bring \( x \) to \( x = 0 \) in a single discrete-time step. However, since \( T \) is by assumption small, a control gain of \( \frac{1}{|I|} \) may be physically impossible to implement. Therefore set \( K_k = \frac{1}{\alpha T} \), where \( \alpha > 1 \), so that the control \( u_k = -K_kI\hat{x}(t_k) \) does not exceed actuator limits.

We will now show that the tracking system described by (16), with \( A \) defined as above, is discrete-time input to state stable (ISS); that is, the state \( x \) converges to a small neighbourhood of the centre of C1’s FOV. The size of this neighbourhood is determined by the magnitude of the estimation error in the position and velocity of the target.

For \( \xi \in \mathcal{X} \), it can be shown (through simulation) that error due to imperfect knowledge of the state \( x(t_k) \) in calculating matrix \( \hat{G} \) is in fact negligible and may be absorbed into the estimation error of \( x(t_{k+1}) \). Then (16) can be rewritten into the following perturbed equations:

\[
x(t_{k+1}) = (I + AT)x(t_k) + \hat{u}_k \quad (17)
\]

\[
u_k = AT\xi_k + \xi^F_k T \quad (18)
\]

For any small fixed \( T \), this can be interpreted as a discrete-time plant with open-loop dynamics described by \( x(t_{k+1}) = (I + AT)x(t_k) \), and a bounded stochastic input \( \hat{u} = AT\xi_k + \xi^F_k T \). Note that \( (I + AT) \) is a Schur matrix. It is a well-known result that a stable linear system with bounded input is ISS (See, for example, [14]). We will show that the ISS property holds even under long measurement times, and calculate an upper bound on the expected tracking performance of the system.

We define the following quadratic Lyapunov function:

\[
V(x(t)) = \frac{1}{2}x(t)'x(t) \quad (19)
\]

In the following, the abridged notation \( V_k \) is used for \( V(x(t_k)) \), and \( V_k \) for \( V(x(t_k)) \). We are interested in the behavior of our system at the measurement instants, \( \{t_k, t_{k+m}, t_{k+2m}, \ldots \} \); we want to estimate the upper bound for the steady state value of the Lyapunov function, which is used to measure tracking performance.

The following two lemmas are used repeatedly in the proofs in this paper:

**Lemma 3.1:** Given a quadratic polynomial

\[
p(x) = -ax^2 + bx + c
\]

with \( a, b, c \in \mathbb{R}^+ \), \( a \neq 0 \), and a non-zero affine function

\[
g(x) = dx + f
\]

with \( d, f \in \mathbb{R}^+ \), there exists some \( C > 0 \) such that

\[
(p + q)(x) < 0 \quad \text{for all} \quad x > C
\]
Proof: Let

\[
C = \frac{(b + d) + \sqrt{(b + d)^2 + 4a(c + f)}}{2a} > 0
\]

Then, \((p + q)(C) = 0\), and \(\frac{d}{dx}(p + q) = (-2ax + (b + d))\) is strictly negative for all \(x \geq C\). Therefore for \(x > C\), \((p + q)(x) < 0\).

Lemma 3.2: Given an inequality \(r(x) \leq s(x)\) for all \(x > 0\), with

\[
s(x) = ax^2 + bx + c
\]

where \(a, b, c \in \mathbb{R}^+\), \(a \neq 0\), and \(s(x) = \phi = \Phi\) for some \(\phi > 0\), it follows that \(r(x) \leq \Phi\) for all \(0 < x \leq \phi\).

Proof: For all \(x > 0\), \(\frac{d}{dx}s(x) = 2ax + b > 0\). Therefore, \(s(x)\) is a strictly increasing function of \(x\) for all \(x > 0\), and it follows that \(r(x) \leq s(x) \leq \Phi\) for all \(0 < x \leq \phi\).

We now turn to our camera tracking system:

Lemma 3.3: For the system defined by (17)-(18) and discrete ISS Lyapunov function (19). \(V_{k+m} - V_k < 0\) whenever \(\|x_c(t_j)\| > h\) for some constant \(h\).

Proof: The difference \(V_{k+m} - V_k\) can be written as

\[
V_{k+m} - \hat{V}_{k+m} + \hat{V}_{k+m} - V_k
\]

and \(V_{k+m} - \hat{V}_{k+m}\) and \(\hat{V}_{k+m} - V_k\) can then be bounded individually.

Begin by finding an upper bound on \(\hat{V}_{k+m} - V_k\). First note that, for all \(i, j\), we can write:

\[
V_j - V_i = \frac{1}{2}x(t_j)'x(t_j) - \frac{1}{2}x(t_i)'x(t_i)
= x(t_i)'(x(t_j) - x(t_i)) + \frac{1}{2}(x(t_j) - x(t_i))'(x(t_j) - x(t_i))
\]

Using this form, we see that \(\hat{V}_{k+1} - V_k\) satisfies the following equation:

\[
\hat{V}_{k+1} - V_k = x(t_k)'(-K_T\hat{x}(t_k) + (F_k - \hat{F}_k)T)
+ \frac{1}{2}\left\| -K_T\hat{x}(t_k) + (F_k - \hat{F}_k)T \right\|^2
= x(t_k)'\left(\frac{-\hat{x}(t_k)}{\alpha} + \xi_k^F T\right)
+ \frac{1}{2}\left\| \frac{-\hat{x}(t_k)}{\alpha} + \xi_k^F T \right\|^2
= x(t_k)'\left(-x(t_k) + \frac{\xi_k}{\alpha} + \xi_k^F T\right)
+ \frac{1}{2}\left\| x(t_k) + \frac{\xi_k}{\alpha} + \xi_k^F T \right\|^2
= \frac{1 - 1}{\alpha}x(t_k)'x(t_k) + \frac{1}{\alpha}x(t_k)'\xi_k + x(t_k)'\xi_k^F T
+ \frac{1}{2}\frac{\xi_k}{\alpha} + \xi_k^F T)'(\frac{\xi_k}{\alpha} + \xi_k^F T)
= \left(\frac{1}{2\alpha^2} - \frac{1}{\alpha}\right)\|x(t_k)\|^2
+ \left(1 + \frac{1}{\alpha}\right)2LT^q\|x(t_k)\| + \left(1 + \frac{1}{\alpha}\right)^2L^2T^2q
\]

Applying the requirement that \(\xi_k \in \mathcal{X}\) (that is, \(\|\xi_k\| \leq LT^q\)) and \(\xi_k^F \in \mathcal{X}^F\) (that is, \(\|\xi_k^F\| \leq LT^{-1}\)) allows us to find an upper bound on \(\hat{V}(x(t_{k+1})) - V(x(t_k))\) as follows:

\[
\hat{V}_{k+1} - V_k \leq \frac{1}{2}\left(\frac{2}{\alpha} + \frac{1}{\alpha^2}\right)\|x(t_k)\|^2
+ \left(1 + \frac{1}{\alpha}\right)^2LT^q\|x(t_k)\| + \left(1 + \frac{1}{\alpha}\right)^2L^2T^2q
\]

So that (using \(V = \frac{1}{2}x'x\)):

\[
\hat{V}_{k+1} \leq (1 - 1/\alpha)^2V_k + (1 + 1/\alpha)^2LT^q\|x(t_k)\|
+ \frac{1}{2}(1 + 1/\alpha)^2L^2T^2q
\]

(20)

Since at each control time \(t_{k+j}\) (where \(j < m\)) we assume that the estimated value of the state is in fact exact, we can apply (20) recursively to get the following inequality:

\[
\hat{V}_{k+m} \leq (1 - 1/\alpha)^{2m}V_k
+ \frac{1 - (1 - 1/\alpha)^{2m}}{2\alpha - 1/\alpha^2}((1 + 1/\alpha)^2LT^q\|x(t_k)\|
+ \frac{1}{2}(1 + 1/\alpha)^2L^2T^2q)
\]

From which we can see that \(\hat{V}_{k+m} - V_k\) is bounded from
above by:

\[ \dot{V}_{k+m} - V_k \leq -\frac{1}{2} \mathbb{I} (1 - 1/\alpha)^{2m} \| x(t_k) \|^2 + \frac{1}{2} \mathbb{I} (1 - 1/\alpha)^{2m} - (1 + 1/\alpha)^2 \lambda L T^q \| x(t_k) \| + \frac{1}{2} (1 + 1/\alpha)^2 L^2 T^q \]  

(21)

We next use the fact that \( \dot{x}(t_{k+1}) = (I + AT)x(t_k) + AT \xi_k + T \xi_k^F \) to rewrite \( x(t_{k+1}) \) in terms of \( x(t_k) \) (assuming that measurements are made at times \( t_k \) and \( t_{k+1} \)). First, note that

\[ \dot{x}(t_{k+1}) = (I + AT)x_k + AT \xi_k + T \xi_k^F \]  

(24)

We next use the fact that \( \dot{x}(t_{j+1}) = (I + AT)x(t_j) + AT \xi_j + T \xi_j^F \) to rewrite \( x(t_{k+1}) \) in terms of \( x(t_k) \) (assuming that measurements are made at times \( t_k \) and \( t_{k+1} \)). First, note that

\[ \dot{x}(t_{k+1}) = (I + AT)x_k + AT \xi_k + T \xi_k^F \]  

(24)

Since no measurement is available at time \( t_{k+1} \), we assume that \( x(t_{k+1}) = \hat{x}(t_{k+1}) \); therefore \( \xi_{k+1} = 0 \) and \( \xi_{k+1}^F = 0 \). Therefore, applying the above equation to \( x(t_{k+1}) \) we get:

\[ \dot{x}(t_{k+1}) = (I + AT)x_k + AT \xi_k + T \xi_k^F \]  

(24)

Again, since there are no new measurements available, we define \( x(t_{k+2}) = \dot{x}(t_{k+2}) \). Applying the above equation recursively, we get:

\[ \dot{x}(t_{k+m}) = (I + AT)^m x(t_k) + (I + AT)^{m-1} AT \xi_k + (I + AT)^{m-1} T \xi_k^F \]  

(25)

\[ \text{from which it follows that: ***correct the following to match changes made above (added a factor of AT in the } \xi \text{ term in above equation for } \dot{x}(t_{k+m})^* \]  

\[ V_{k+m} - \hat{V}_{k+m} = (I + AT)^m x(t_k) + (I + AT)^{m-1} \xi_k \]  

(26)

\[ \leq (1 - 1/\alpha)^m L (mT)^q \| x(t_k) \| + (2(1 - 1/\alpha)^m - 1/2) m^2 q) L^2 T^2 q \]  

(27)

Eq. (21) is a quadratic polynomial in \( \| x(t_k) \| \) with negative leading-order coefficient, and (27) is an affine function in \( \| x(t_k) \| \). Applying Lemma 3.1, we see that we can define \( h \) such that \( V_{k+m} - V_k < 0 \) whenever \( \| x(t_k) \| > h \) where:

\[ h = \frac{(b + d) + \sqrt{(b + d)^2 + 4a(c + f)}}{2a} \]  

(28)

whith

\[ a = \frac{1}{2} (1 - (1 - 1/\alpha)^2m) \]  

(29)

\[ b = \frac{1}{2} (1 - (1 - 1/\alpha)^2m)(1 + 1/\alpha)^2 L T^q \]  

(30)

\[ c = \frac{1}{2} (1 - (1 - 1/\alpha)^2m)^{-1} \frac{1 + 1/\alpha)^2 L^2 T^2 q}{2} \]  

(32)

\[ f = (2(1 - 1/\alpha)^m - 1/2) m^2 q) L^2 T^2 q \]  

(33)

The following theorem establishes the invariance of the neighborhood defined by \( V \leq \frac{1}{2} h^2 \).

**Theorem 3.1**: Consider the closed loop system dynamics given by (13)-(15), with control interval \( T \) and measurement interval \( mT \). Assume that position error \( \xi \) belongs to \( X \) and velocity error \( \xi^F \) to \( X^F \). The control gain is a constant, \( \frac{1}{\alpha} \). Then as \( k \to \infty \), \( \| x(t_k) \| \) is bounded above by \( h \), which is defined as in eq. (28)-(33).

**Proof**: Consider the function \( W_i \) defined as \( W_{k+m} = \max_{i \in \{k, k+m\}} V_i \). Consider a time \( t_k \) such that \( W_k \geq \frac{1}{2} h^2 \). Then, by Lemma 3.1, \( V_{k+m} - V_k < 0 \). Therefore, we will have \( V_{k+m} < V_k \). Since increases in \( V_i \) occur only at measurement times, this also means that \( W_{k+m} < W_k \). Therefore, as \( k \) increases, the value of \( W_k \) will decrease until \( W_k \leq \frac{1}{2} h^2 \).

Now consider \( W_k \leq \frac{1}{2} h^2 \). We will show that the value of \( W_{k+m} \) is bounded above by \( \frac{1}{2} h^2 \), so that once \( V_k \) is inside the set \( \{ V_k : V_k < \frac{1}{2} h^2 \} \), it stays there for all time \( t \geq t_k \). First, using the fact that \( V_k = \frac{1}{2} \| x(t_k) \|^2 \), we rewrite the sum of (21) and (27) as:

\[ V_{k+m} \leq \frac{1}{2} (1 - 1/\alpha)^2 m \| x(t_k) \|^2 + (b + d) \| x(t_k) \| + (c + f) \]  

(34)

\[ \| x(t_{k+m}) \|^2 + \frac{\bar{a} \| x(t_k) \|^2}{2} + \bar{b} \| x(t_k) \| + \bar{c} \]  

(35)

where \( \bar{a}, \bar{b}, \bar{c} > 0 \). The upper bound on \( W_{k+m} \) is a positive quadratic in \( \| x(t_k) \| \) with value \( \frac{1}{2} h^2 \) at \( x(t_k) = h \). By Lemma 3.2, \( V_{k+m} \leq \frac{1}{2} h^2 \) for all \( \| x(t_k) \| \leq h \). Since increases in \( V_i \) for \( i \in \{k, k+m\} \) occur only at the measurement times \( t_k \), \( V_{k+m} \leq \frac{1}{2} h^2 \) means that \( V_i \leq \frac{1}{2} h^2 \) for all \( i \in \{k, k+m\} \), which implies that \( W_{k+m} \leq \frac{1}{2} h^2 \). Therefore \( W_{k+m} \) is bounded above by \( \frac{1}{2} h^2 \) for all \( t \geq t_k \).

Therefore, as \( k \to \infty \), \( W_k \leq \frac{1}{2} h^2 \), where \( h \) is given by (28)-(33).

The bound \( h \) on the steady-state tracking error is directly proportional to the control interval \( T \) and the measurement interval \( mT \). In order to achieve the exact tracking i.e. \( x(t_k) = 0 \) for all \( k \), no control or measurement delay should be tolerated. Therefore, under time delay, the best we can hope is to stabilize the target in a small neighborhood of the center of the FOV. The size of this neighborhood relates to the maximum possible delay through (28)-(33). If the delay is longer, the size of the neighborhood is bigger which implies that the tracking performance will be degraded.
IV. TRACKING MULTIPLE TARGETS WITH ASYNCHRONICITY

Now consider the case where multiple targets are monitored by the camera C1, and one or more of them may be occluded. If occlusion does occur, camera C1 communicates with other cameras to obtain estimates for the states of the occluded targets. In order to reduce networking traffic, we assume that the communication between cameras happens infrequently, so that not all estimates made by these other cameras will be sent to the requesting camera C1. For example, if a camera has estimates available every 10ms, the camera may choose to send the estimates every 50ms. Therefore, at C1, estimates for hidden targets will be available less frequently than estimates for targets which have clear line of sight. We want to find out how frequently communication must happen to guarantee specified performance levels on target tracking.

![Image](fig.2.png)

**Fig. 2. Two-target tracking system with one occluded target**

We analyze a simple case where we assume that there are two point targets (\(N = 2\)), of which one (say, target 1) is occluded (see fig. 2). Camera C1 generates updates for the position of any unoccluded target with interval T. The estimate of the state of target 1 is communicated to C1 at time intervals of length \(mT\), where integer \(m > 1\). Suppose that these estimates are available at times \(t_k, t_{k+m}, t_{k+2m}, \ldots\) on the interval \([t_k, t_{k+m})\), estimates for the unblocked target (target 2) are updated \(m\) times and the estimate for the occluded target (target 1) is updated only once. This causes asynchronicity in the discretized system

\[
x_c(t_{k+1}) = x_c(t_k) + G_{c,k} u_k T + F_{c,k} T
\]

where

\[
x_c(t_k) = \frac{1}{N} \sum_{i=1}^{N} x_i(t_k), \quad G_{c,k} = G_c(t_k), \quad F_{c,k} = \frac{1}{N} \sum_{i=1}^{N} B_i(t_k).
\]

Consider the time interval \([t_k, t_{k+m})\). As in the single-target case, we use a simple linear predictor to construct position estimates for \(x_c\):

\[
\hat{x}_c(t_{j+1}) = x_c(t_j) + G_{c,j} u_j T + F_{c,j} T
\]

\[
\hat{F}_{c,j+1} = \hat{F}_{c,j}
\]

\[
u_{c,j} = \hat{G}_{c,j}^{-1} (A \hat{x}_c(t_j) - \hat{F}_{c,j})
\]

When no new estimate for target 1 is available, we assume that the estimated values for \(x_c\) and \(F_c\) are exact. As before, we can rewrite the state equations in the approximate form:

\[
\hat{x}_c(t_{k+1}) = (I + AT)x_c(t_k) + \hat{u}_k
\]

\[
u_k = AT \xi_{c,k} + \xi_{c,k} F T
\]

which is a perturbed linear system driven by stochastic noise inputs \(\xi_{c,k}\) and \(\xi_{c,k}\). This time, however, unlike the single-target case, each of the noise inputs has two distinct components, corresponding to the errors on the states of targets 1 and 2: \(\xi_{c,k} = \frac{1}{2}(\tilde{\xi}_{1,k} + \tilde{\xi}_{2,k})\) and \(\xi_{c,k} = \frac{1}{2}(\tilde{\xi}_{c,k} + \tilde{\xi}_{c,k})\). As in the single-target case, we assume that the error in estimated position for each of the targets is \(\xi_{c,j} \in \mathcal{X} = \left\{ \phi : \| \phi \| \leq T (\Delta t)^q \right\}, \forall j\) and error in estimated velocity is \(\xi_{c,j} \in \mathcal{V} = \left\{ \psi : \| \psi \| \leq T (\Delta t)^{q-1} \right\}, \forall j, i = 1, 2\). Here \(L \in \mathbb{R}^+\) and \(q \in \mathbb{Z}^+\) are known, and \(\Delta T\) is the time since the last available measurement. The matrix \(A\) is selected to be \(A = -K_k T\) with \(K = 1/\alpha T\).

**Lemma 4.1:** Consider a camera tracking system with dynamics described by eq. (40)-(41). The inter-control input interval is \(T\) and the inter-measurement intervals are \(mT\) for target 1 and \(T\) for target 2. The matrix \(A\) and bounds on the estimates of states of the targets are defined as above. Then the system is discrete-time input to state stable (ISS) and admits a quadratic lyapunov function.

**Proof:** The updates to the state of target 1 and 2 are available with \(\Delta t = mT\) and \(T\), respectively. The maximum position error we may expect at any measurement time, therefore, is \(\xi_c = \frac{1}{2} L((mT)^q + T^q)\); the maximum velocity error is \(\xi^F = \frac{1}{2} L((mT)^{q-1} + T^{q-1})\). Both \(\xi_c\) and \(\xi^F\) are bounded for any \(T < \infty\).

With \(A\) defined as \(A = -\frac{1}{\alpha T} I\), \(I + AT\) is a Schur matrix. Therefore, as in the proof of Lemma 4.1, our system admits an ISS Lyapunov function, and is necessarily ISS.

Input to state stability of the system (40)-(41) is guaranteed by the above Lemma; the state will converge to a bounded neighbourhood of the origin, which can be estimated by constructing an explicit discrete ISS Lyapunov function \(V\) for the system and finding values of \(x_c(t_k)\) such that \(V_{k+m} - V_k < 0\). Estimating the size of the neighbourhood, however, is less straightforward than before.

Consider a simple quadratic ISS Lyapunov function for the system:

\[
V_{c,k} = \frac{1}{2} x_c(t_k)^T x_c(t_k)
\]

We will analyze the behavior of this function over a time interval \((t_k, t_{k+m})\), where we assume that measurements of the state of target 1 are available at times \(t_k, t_{k+m}\). Similar to the single-target case, \(V_{c,k+m} - V_{c,k}\) can be as:

\[
V_{c,k+m} - V_{c,k} = V_{c,k+m} + \sum_{j=k+1}^{k+m-1} (-V_{c,j} + V_{c,j}) - V_{c,k}
\]

Thus, to guarantee \(V_{c,k+m} - V_{c,k} < 0\), it is sufficient to have \(V_{c,j+1} - V_{c,j} < 0\) for all \(j \in \{k, k+1, \ldots, k+m - 1\}\).

**Lemma 4.2:** For the system defined by (40)-(41) and discrete ISS Lyapunov function (42), for all \(\| x_c(t_j) \| > h_1\),

\[
h_1 = \frac{\alpha^2 L T^q (1 + (1 + 1/\alpha)^2 + 2\sqrt{1 + 4/\alpha + 2/\alpha^2} + 1/\alpha^2)}{2(\alpha - 1)}
\]

we have \(V_{c,j+1} - V_{c,j} < 0\) for all \(j \in \{k, k+1, \ldots, k+m - 1\}\).
Proof: For all times \( t \in \{ t_{k+1}, t_{k+2}, \ldots, t_{k+m-1} \} \), the error in the estimate of state of target 1 is 0. Therefore, at these times, the only error contribution in estimation of the state \( x_c \) is error in the estimate of the state of target 2, \( \xi_{2,k} \) and \( \xi_{F,k} \). This in mind we write \( V_{c,j+1} - V_{c,j} \) as:

\[
V_{c,j+1} - V_{c,j} = V_{c,j+1} - \hat{V}_{c,j+1} + \hat{V}_{c,j+1} - V_{c,j} \tag{43}
\]

We analyze \( V_{c,j+1} - \hat{V}_{c,j+1} \) and \( \hat{V}_{c,j+1} - V_{c,j} \) separately.

First, note that since the error bound and control law are identical to the single-target case, \( \hat{V}_{c,j+1} - V_{c,j} \) is bounded just as in the single-target case:

\[
\begin{aligned}
\hat{V}_{c,j+1} - V_{c,j} &\leq -\frac{1}{2} \left( \frac{2}{\alpha} - \frac{1}{\alpha^2} \right) \| x_c(t_k) \|^2 \\
&\quad + \frac{1}{2} \left( \frac{1}{\alpha} + 1 \right) L^T q \| x_c(t_k) \| + \frac{1}{8} \left( \frac{1}{\alpha} + 1 \right) ^2 L^2 T^{2q}
\end{aligned}
\]

(44)

Similarly, \( V_{c,j+1} - \hat{V}_{c,j+1} \) can be written as:

\[
\begin{aligned}
V_{c,j+1} - \hat{V}_{c,j+1} &\leq \frac{2}{\alpha} \| x_c(t_{j+1}) \| + \frac{3}{8} L^2 T^{2q}
\end{aligned}
\]

(45)

Applying Lemma 3.1 to (44) and (46), we see that \( V_{c,j+1} - V_{c,j} < 0 \) whenever \( \| x_c(t) \| > h_1 \), where \( h_1 \) is given by:

\[
h_1 = \frac{\alpha^2 L^T q \left( 1 + (1 + 1/\alpha)^2 \right) + 2 \sqrt{1 + 4/\alpha^2 + 2/\alpha^2 + 1/\alpha^3}}{2(2\alpha - 1)}
\]

(47)

Lemma 4.3: For the system defined by (40)-(41) and discrete ISS Lyapunov function (42),

\[
\max\{ V_{c,k+1}, \ldots, V_{c,k+m-1} \} \leq \max\{ V_{c,k}, \frac{1}{2} h_1^2 \}
\]

where \( h_1 \) is defined as in (47).

Proof: First suppose that \( x_c(t_k) > h_1 \), corresponding to \( V_{c,k} > \frac{1}{2} h_1^2 \). By Lemma 4.2, this means that \( V_{c,k+1} - V_{c,k} < 0 \), so that \( V_{c,k+1} < V_{c,k} \). By induction, \( V_{c,j+1} < V_{c,j} \) holds for each \( j \in \{ k+1, \ldots, k+m-1 \} \) so long as \( x_c(t_j) > h_1 \).

Now suppose that for some \( j \), \( \| x_c(t_k) \| \leq h_1 \). Then, adding equations (44) and (46) for time \( t_j \), we get:

\[
\begin{aligned}
V_{c,j+1} - V_{c,j} &\leq -\frac{1}{2} \left( \frac{2}{\alpha} - \frac{1}{\alpha^2} \right) \| x_c(t_k) \|^2 \\
&\quad + \frac{1}{2} \left( \frac{1}{\alpha} + 1 \right) L^T q \| x_c(t_k) \| + \frac{1}{8} \left( \frac{1}{\alpha} + 1 \right) ^2 L^2 T^{2q}
\end{aligned}
\]

(48)

Adding \( V_{c,j} = \frac{1}{2} \| x_c(t_j) \| \) to both sides of the inequality, we see that:

\[
\begin{aligned}
V_{c,j+1} &\leq \frac{1}{2} \left( 1 - \frac{1}{\alpha^2} \right) \| x_c(t_j) \|^2 \\
&\quad + \left( \frac{1}{\alpha} + 1 \right) L^T q \| x_c(t_j) \| + \frac{1}{8} \left( \frac{1}{\alpha} + 1 \right) ^2 L^2 T^{2q}
\end{aligned}
\]

(49)

where \( \tilde{a}, \tilde{b}, \tilde{c} \) are strictly greater than 0 and \( \tilde{a} h_1^2 + \tilde{b} h_1 + \tilde{c} = \frac{1}{2} h_1^2 \). By Lemma 3.2, therefore, \( V_{c,j+1} < \frac{1}{2} h_1^2 \) for all \( \| x_c(t_j) \| < h_1 \).

Overall, for all initial values \( V_{c,k} \),

\[
\max\{ V_{c,k+1}, \ldots, V_{c,k+m} \} \leq \max\{ V_{c,k}, \frac{1}{2} h_1^2 \}
\]

Lemma 4.4: For the system defined by (40)-(41) and discrete ISS Lyapunov function (42), there exists an \( h_2 > h_1 \) such that \( V_{c,k+m} - V_{c,k+m-1} < 0 \) for all \( \| x_c(t_{k+m-1}) \| > h_2 \).

Proof: At time \( t_{k+m} \), estimates of the states of both targets are available, so that the error in position is given by \( \xi_{c,k+m} = \frac{1}{2}(\xi_{1,k+m} + \xi_{2,k+m}) \), where \( \xi_{1,k+m} \in \mathcal{X} \) with \( \Delta t = mT \) and \( \xi_{2,k+m} \in \mathcal{X} \) with \( \Delta t = T \). The error in velocity is given by \( \xi_{F,k+m} = \frac{1}{2}(\xi_{F1,k+m} + \xi_{F2,k+m}) \), where \( \xi_{F1,k+m} \in \mathcal{X} \) with \( \Delta t = mT \) and \( \xi_{F2,k+m} \in \mathcal{X} \) with \( \Delta t = T \). Thus, we can say that \( \xi_{c,k+m} \in \{ \phi : \| \phi \| \leq \frac{1}{2} (1 + m^q) L^T q \} \) and \( \xi_{F,k+m} \in \{ \psi : \| \psi \| \leq \frac{1}{2} (1 + m^q) L^T q \} \).

We write \( V_{c,k+m} - V_{c,k,m-1} \) as:

\[
\begin{aligned}
V_{c,k+m} - V_{c,k,m-1} &\leq V_{c,k+m} - \hat{V}_{c,k+m} + \hat{V}_{c,k+m} - V_{c,k,m-1}
\end{aligned}
\]

(50)

and analyze \( V_{c,k+m} - \hat{V}_{c,k,m} \) and \( \hat{V}_{c,k,m} - V_{c,k,m-1} \) separately.

It can be shown that:

\[
\begin{aligned}
\hat{V}_{c,k,m} - V_{c,k,m-1} &\leq -\frac{1}{2} \left( \frac{2}{\alpha} - \frac{1}{\alpha^2} \right) \| x_c(t_{k+m-1}) \|^2 \\
&\quad + \frac{1}{2} \left( \frac{1}{\alpha} + 1 \right) L^T q \| x_c(t_{k+m-1}) \| + \frac{1}{8} \left( \frac{1}{\alpha} + 1 \right) ^2 L^2 T^{2q}
\end{aligned}
\]

(51)

Then, we can find a bound on \( V_{c,k+m} - \hat{V}_{c,k,m} \) as:

\[
\begin{aligned}
V_{c,k+m} - \hat{V}_{c,k,m} &\leq \frac{1}{2} (1 + m^q) L^T q \| x_c(t_{k+m}) \| + \frac{3}{8} (1 + m^q) L^2 T^{2q}
\end{aligned}
\]

(52)

Adding together equations (51) and (52) and applying Lemma 3.1, we see that \( V_{c,k+m} - V_{c,k,m-1} < 0 \) whenever \( \| x(t_{k+m}) \| > h_2 \), where \( h_2 \) is given by:

\[
h_2 = \frac{\alpha^2 L^T q (1 + (1 + 1/\alpha)^2 + 2\sqrt{H})}{2(2\alpha - 1)}
\]

(53)
where

\[ H = 1 + 4/\alpha + 2/\alpha^2 + 1/\alpha^3 + m^2 + (4 + 6/\alpha + 1/\alpha^2)m^q \]

(54)

Note that \( h_2 > h_1 \).

Lemma 4.5: For the system (40)-(41) and discrete ISS Lyapunov function (42),

\[ \max\{ V_{c,k+1, \ldots, c,k+m} \} \leq \max\{ V_{c,k}, \frac{1}{2} h_2^2 \} \]

where \( h_2 \) is defined as in (53).

Proof: By Lemma 4.3, we already know that

\[ \max\{ V_{c,k+1, \ldots, c,k+m-1} \} \leq \max\{ V_{c,k}, \frac{1}{2} h_1^2 \} \]

where \( h_1 < h_2 \), so that \( \frac{1}{2} h_1^2 < \frac{1}{2} h_2^2 \).

Suppose that \( V_{c,k+m-1} > \frac{1}{2} h_2^2 \). This means that \( V_{c,k+m-1} \geq \frac{1}{2} h_2^2 \), and therefore, perforce, \( V_{c,k+m-1} < V_c \).

In this case, by Lemma 4.4, \( V_{c,k+m-1} - V_{c,k+m-1} < 0 \), so that \( V_{c,k+m-1} < V_{c,k+m-1} \), and we have established that

\[ \max\{ V_{c,k+1, \ldots, c,k+m-1} \} = V_{c,k} \leq \max\{ V_{c,k}, \frac{1}{2} h_2^2 \} \]

Now suppose that \( V_{c,k+m-1} \leq \frac{1}{2} h_2^2 \). Adding equations (51) and (52) for time \( t_{k+m-1} \), we get:

\[
\begin{align*}
V_{c,k+m} & \leq \frac{1}{2} \left( \frac{2}{\alpha} - \frac{1}{\alpha^2} \right) \| x_c(t_{k+m}) \|^2 \\
& \quad + \frac{1}{2} \left( (1 + 1/\alpha^2) + 1 + m^q \right) LT^q \| x_c(t_k) \|^2 \\
& \quad + \frac{1}{8} \left( (1 + 1/\alpha^2) + 3(1 + m^q) \right) L^2 T^{2q} \\
= & \hat{a} \| x_c(t_{k+m}) \|^2 + \hat{b} \| x_c(t_{k+m-1}) \| + \hat{c}
\end{align*}
\]

(55)

Then, adding \( V_{c,m+k-1} = \frac{1}{2} \| x_c(t_{m+k-1}) \| \) to both sides of the inequality, we see that:

\[
\begin{align*}
V_{c,k+m} & \leq \frac{1}{2} \left( 1 - \frac{1}{\alpha} \right) \| x_c(t_{k+m-1}) \|^2 \\
& \quad + \left( \frac{1}{\alpha} + 1 \right) LT^q \| x_c(t_{k+m-1}) \| + \frac{1}{2} L^2 T^{2q} \\
& = \hat{a} \| x_c(t_{k+m-1}) \|^2 + \hat{b} \| x_c(t_{k+m-1}) \| + \hat{c}
\end{align*}
\]

(56)

where \( \hat{a}, \hat{b}, \hat{c} \) are strictly greater than zero, and \( \hat{a}h_2^2 + \hat{b}h_2^2 + \hat{c} = \frac{1}{2} h_2^2 \). Therefore, by Lemma 3.2, \( V_{c,k+m} < \frac{1}{2} h_2^2 \), for all \( \| x_c(t_{k+m-1}) \| \leq h_2 \).

Since it is known that

\[
\max\{ V_{c,k+1, \ldots, c,k+m-1} \} \leq \max\{ V_{c,k}, \frac{1}{2} h_2^2 \} \]

(57)

And we have shown that for arbitrary \( V_{c,k+m-1} \), \( V_{c,k+m} \leq \max\{ V_{c,k}, \frac{1}{2} h_2^2 \} \), it follows that

\[
\max\{ V_{c,k+1, \ldots, c,k+m} \} \leq \max\{ V_{c,k}, \frac{1}{2} h_2^2 \}
\]

Lemma (4.5) implies that as \( k \to \infty \), \( x_c(t_k) \) converges to a neighbourhood of size \( h_2 \) around the center of C1’s FOV.

This upper bound on the size of the neighbourhood will tend to be conservative, as we have always assumed a worst-case estimation error for the state of each of the targets. However, it is a useful metric for estimating the performance of the networked camera tracking system given computation and communication delay and synchronicity.

V. SIMULATION

The above camera tracking system is simulated in Matlab for both the single-target and the two-target asynchronous measurement case. Arbitrary deterministic 2-dimensional trajectories are assigned to the targets. It was assumed that the camera system can observe the location of each object with additive Gaussian white noise.

Given the nonlinear dynamics of the targets and measurement noise, it is desirable to use a short-memory filter to estimate the state \( x \) from a given observation. A polynomial predictor (implemented using Matlab polyfit function) was used to estimate the state at each control instant. The ten most recent data points were used to find a least-squares second-order polynomial function for \( x_1 \) and \( x_2 \) over time.

The simulation was run with \( T = 10^{-3} \) sec and \( m \) varying between 1 (target not occluded) to 5 (target occluded). A section of the target space was designated “brush area”, with targets becoming occluded whenever they entered this area. As illustrated in fig. 3, the estimates of the size of the neighborhood of the center of the camera’s FOV in the previous sections tend to be much higher than the neighborhoods obtained in simulation. This is not surprising, since we have always assumed a worst-case scenario, and calculated bounds on \( V(x(t)) \) corresponding to maximum possible error at every measurement instant. It may be observed, however, that the size of the neighborhood to which \( x \) converges is dependent on \( mT \), as expected from our model and calculations.

REFERENCES


