Robust Control of Horizontal Formation Dynamics for Autonomous Underwater Vehicles

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Abstract— **This paper presents a novel robust controller design for formation control of autonomous underwater vehicles (AUVs). We consider a nonlinear three-degree-of-freedom dynamic model for the horizontal motion of each AUV. By using the Jacobi transform, the horizontal dynamics of AUVs are explicitly expressed as dynamics for formation shape and formation center, and are further decoupled by feedback control. We treat the coupling terms as perturbations to the decoupled system. An H_∞ state feedback controller is designed to achieve robust stability of the closed loop formation and translation dynamics. By incorporating an orientation controller, the formation shape under control converges and the formation center tracks a desired trajectory simultaneously. Simulation results demonstrate the effectiveness of the controllers.**

I. INTRODUCTION

One of the major difficulties of formation control for AUVs is that the collected dynamics of all vehicles are more complex than the non-trivial single vehicle dynamics. A common practice in some of the existing results for formation control is to simplify the motion dynamics of an individual vehicle or robot to a second-order particle model [1]–[8], but formation control becomes more challenging if more practical and complex dynamics are concerned. Various methods have been developed to answer this challenge. A leader-follower formation control scheme for autonomous helicopters is investigated in [9] by applying the sliding-mode controller design method, where a 6 DOF dynamic model is considered. In [10], a dynamic model of the AUV ODIN [11] is used to design a proportional-derivative controller for formation control. A 3 DOF horizontal model for AUVs is used in [12] and [13]. In [12], the model decouples sway and yaw motion. A virtual vehicle is employed to provide a reference trajectory and velocity for the followers with their tracking controllers designed using the backstepping method. In [13], the horizontal dynamic model of a torpedo type AUV is described using a general nonlinear mapping, and formation controllers are designed based on artificial potential functions. A cross-track control scheme based on the Line of Sight (LOS) guidance law is presented to make the AUVs follow a given straight line and constitute a desired formation in [14], where a 5 DOF dynamic model with independent control inputs in surge, pitch and yaw is considered. Similar approaches are extended to surface vessels described by a 3 DOF dynamic model whose surge dynamics are decoupled from the steering dynamics [15]. A cooperative controller based on discrete time Kuramoto models is designed to integrate communication and control for multiple vehicles [16]. Experimental results on University of Washington Fin-actuated Autonomous Underwater System(UMMFAUS) are introduced in [17].

The reviewed existing methods design formation controllers for the collected dynamics directly. The complex vehicle dynamics lead to results that are difficult to be justified theoretically. In authors' paper [18], we express the formation dynamics as a deformable body by using the Jacobi coordinates that have been previously applied to formation control for particles in Zhang's works [4]–[7]. We have shown that due to the symmetry of the ODINs, the formation dynamics are linear and naturally decoupled from the translation dynamics, similar to particle formation systems. This allows linear feedback controllers to stabilize the formation system. In this paper, our main contribution is the generalization of the formation control approach in [18] from the special model of the ODIN to more realistic AUVs controlled by multiple thrusters. Due to lack of symmetry of a generic AUV, the decoupling between the formation dynamics and the translation dynamics has to be achieved by feedback control. This lack of symmetry is treated as perturbations to a decoupled system. We then follow a robust design approach that results in a H_∞ control law that asymptotically stabilizes the decoupled dynamics, and achieves robust stability for the coupled dynamics.

There exist other results in the literature on robust controller designs for formations of AUVs. In [19], using the linear quadratic gaussian (LQG) design method, steering and speed controllers are designed for a leader-follower formation consisted of three AUVs. The loop transfer recovery (LTR) techniques are used to recover robustness. In [20], the successive Galerkin approximation (SGA) approach is applied to the robust control of a leader-follower formation system of a pair of AUVs, whose formation dynamics are described by a four-input driftless chained form. A reduced order H_∞ controller is designed in [21] for Subzero III, which is a test-bed of control techniques for AUVs. The nonlinear dynamic model of Subzero III is simplified to three decoupled SISO subsystem transfer functions i.e. speed, heading and depth. Comparing to the above approaches, our approach provides explicit descriptions for the perturbations on the formation dynamics and perturbations on the translation dynamics. The complexity of the controller structure is significantly reduced.

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The organization of this paper is as follows. In Section II, the horizontal dynamic equations of a single AUV are reviewed. And the formation dynamics of multiple AUVs through Jacobi transform are derived. In Section III, the formation dynamic model is transformed into a system where the formation shape dynamics and the translation dynamics are decoupled, and the coupling terms are viewed as perturbations to the system. An orientation controller, a robust velocity controller, and a position controller for the formation system are designed in Section IV. Numerical simulation results are given in Section V. The last section contains conclusions.

II. FORMATION DYNAMICS

A. Model of Single AUV

We consider the 3 DOF horizontal motion model of an AUV with port/starboard symmetry (as is common for AUVs) with multiple thrusters. Assuming the center of gravity and the center of mass coincide, we have the following equations [22], [23]:

$$\dot{\eta} = R_b^I(\psi)\nu \tag{1}$$

$$M\dot{\nu} + C(\nu)\nu + D\nu = \tau' \tag{2}$$

where, $\eta = [x, y, \psi]^T$ represents the position and the yaw angle of the vehicle; $\nu = [u, v, r]^T$ is the velocity vector that contains surge, sway and yaw. And, let c represents cos() and s represents sin(), we have

$$\begin{aligned}
&R_b^I(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
&M = \mathrm{diag}\{m - X_{\dot{u}}, m - Y_{\dot{v}}, I_z\}, \\
&C(\nu) = \begin{bmatrix} 0 & 0 & -mv + Y_{\dot{v}}v \\ 0 & 0 & mu - X_{\dot{u}}u \\ mv - Y_{\dot{v}}v & -mu + X_{\dot{u}}u & 0 \end{bmatrix}, \\
&D = \mathrm{diag}\{-X_u - X_{u|u|}|u|, -Y_v - Y_{v|v|}|v|, -N_r - N_{r|r|}|r|\}, \\
&\tau' = \begin{bmatrix} \tau_x & \tau_y & \tau_\psi \end{bmatrix}^T.
\end{aligned} \tag{3}$$

The matrix $R_b^I(\psi)$ is the rotation matrix from the body frame to the inertial frame. M denotes a simplified inertia matrix where the added mass terms are $X_{\dot{u}}$ and $Y_{\dot{v}}$, and an added inertia term is contained in I_z. $C(\nu)$ contains Coriolis and centrifugal force terms, and D is the hydrodynamic damping matrix.The control input vector τ' is composed of surge force τ_x, sway force τ_y, and yaw moment τ_ψ. The terms X_u, Y_v, N_r, $X_{u|u|}$, $Y_{v|v|}$, and $N_{r|r|}$ are hydrodynamic parameters.

In order to make the equations more concise, let $m_{11} = m - X_{\dot{u}}$, $m_{22} = m - Y_{\dot{v}}$, $m_{33} = I_z$, $d_{11} = -X_u - X_{u|u|}|u|$, $d_{22} = -Y_v - Y_{v|v|}|v|$, and $d_{33} = -N_r - N_{r|r|}|r|$. A couple of facts should be noticed. First, m_{11}, m_{22}, and m_{33} are positive for AUVs. Second, even though d_{11}, d_{22}, and d_{33} depend on u, v, and r, they are all non-negative regardless of u, v,and r. These facts will be leveraged to derive the equations of formations and for controller design.

Let $p = [x, y]^T$ and $\gamma = [u, v]^T$. We can rewrite the position and orientation transforms described in equation (1) and (2) as follows [18]:

$$\dot{p} = R(\psi)\gamma \tag{4}$$

$$\dot{\gamma} = M_1^{-1}(\tau - D_1(r, |u|, |v|)\gamma) \tag{5}$$

$$\begin{cases} \dot{\psi} = r \\ \dot{r} = \frac{(m_{11} - m_{22})}{m_{33}}uv - \frac{d_{33}}{m_{33}}r + \frac{1}{m_{33}}\tau_\psi \end{cases} \tag{6}$$

where $R(\psi) = \begin{bmatrix} c\psi & -s\psi \\ s\psi & c\psi \end{bmatrix}$ has the properties that $R^T(\psi)R(\psi) = I$ for all ψ, and $\dot{R}(\psi) = R(\psi)S(\dot{\psi})$ where $S(\dot{\psi}) = \begin{bmatrix} 0 & -\dot{\psi} \\ \dot{\psi} & 0 \end{bmatrix}$ is skew-symmetric. Here $M_1 = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}$, $D_1(r, |u|, |v|) = \begin{bmatrix} d_{11} & -m_{22}r \\ m_{11}r & d_{22} \end{bmatrix}$, and $\tau = \begin{bmatrix} \tau_x & \tau_y \end{bmatrix}^T$.

Taking derivatives on both sides of equation (4) yields

$$\begin{aligned}
\ddot{p} &= \dot{R}(\psi)\gamma + R(\psi)\dot{\gamma} = R(\psi)S(r)\gamma + R(\psi)\dot{\gamma}. \\
&= R(\psi)[S(r) - M_1^{-1}D_1(r, |u|, |v|)]R^{-1}(\psi)\dot{p} \\
&+ R(\psi)M_1^{-1}\tau.
\end{aligned} \tag{7}$$

Define

$$G(\psi, r, |u|, |v|) = R(\psi)[S(r) - M_1^{-1}D_1(r, |u|, |v|)]R^{-1}(\psi) \tag{8}$$

and

$$H(\psi) = R(\psi)M_1^{-1}. \tag{9}$$

Then equation (7) can be rewritten as:

$$\ddot{p} = G(\psi, r, |u|, |v|)\dot{p} + H(\psi)\tau. \tag{10}$$

Equation (10) and the equations of vehicle orientation described by equation (6) are nonlinear equations about state variables $(\psi, r, p, \dot{p})$ with control inputs τ and τ_ψ.

B. Formation Dynamics

The entire formation of N AUVs can be viewed as a deformable body. whose shape and movement can be described using Jacobi vectors [6]. Suppose the positions of the AUVs are described by $p_i = [x_i, y_i]^T, i = 1, 2, ..., N$. Then the Jacobi vectors are defined by a linear transform Φ that produces the following equation:

$$[\rho_1^T, \rho_2^T, ..., \rho_{N-1}^T, q_c]^T = \Phi[p_1^T, p_2^T, ..., p_N^T]^T \tag{11}$$

where ρ_j, $(j = 1, 2, \ldots, N-1)$, are the Jacobi vectors and q_c is the formation center defined by $q_c = \frac{1}{N}\sum_{i=1}^N p_i$. Define the notation $G_i = G(\psi_i, r_i, |u_i|, |v_i|)$ for $i = 1, 2, ..., N$ where ψ_i and r_i are the yaw angle and angular speed, and u_i and v_i are the surge and sway speed of the ith AUV. The second order derivatives of equation (11) are computed as follows:

$$\begin{bmatrix} \ddot{\rho}_1 \\ \vdots \\ \ddot{\rho}_{N-1} \\ \ddot{q}_c \end{bmatrix} = \Phi \begin{bmatrix} \ddot{p}_1 \\ \vdots \\ \ddot{p}_N \end{bmatrix} = \Phi \begin{bmatrix} G_1\dot{p}_1 + H(\psi_1)\tau_1 \\ \vdots \\ G_N\dot{p}_N + H(\psi_N)\tau_N \end{bmatrix}$$
$$= \Phi \begin{bmatrix} G_1 & & \\ & \cdots & \\ & & G_N \end{bmatrix} \Phi^{-1} \begin{bmatrix} \dot{\rho}_1 \\ \vdots \\ \dot{\rho}_{N-1} \\ \dot{q}_c \end{bmatrix}$$
$$+ \Phi \begin{bmatrix} H(\psi_1) & & \\ & \cdots & \\ & & H(\psi_N) \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_N \end{bmatrix}. \tag{12}$$

Define a state vector

$$X = [\rho_1^T, \ldots, \rho_{N-1}^T, q_c^T]^T \tag{13}$$

and let

$$\mathbf{G} = \mathrm{diag}\{G_1, G_2, ..., G_N\}. \tag{14}$$

Then the dynamic equations of the formation can be further written as the following:

$$\ddot{X} = \mathbf{A}([r_i],[\psi_i],[|u_i|],[|v_i|])\dot{X} + \Gamma([\psi_i])U \tag{15}$$

where

$$\mathbf{A}([r_i],[\psi_i],[|u_i|],[|v_i|]) = \Phi\mathbf{G}\Phi^{-1}, \tag{16}$$

$$\Gamma([\psi_i]) = \Phi\mathrm{diag}\{H(\psi_1), ..., H(\psi_N)\}, \tag{17}$$

and

$$U = \begin{bmatrix} \tau_1 & \cdots & \tau_N \end{bmatrix}^T. \tag{18}$$

The block diagonal matrix $\mathbf{G}$ plays an important role in the formation dynamics since it determines whether the formation shape dynamics described by the Jacobi shape vectors ρ_i and the formation center dynamics described by the center vector q_c are decoupled. For each block of $\mathbf{G}$, we have

$$\begin{aligned} G_i &= R(\psi_i)[S(r_i) - M_1^{-1}N(|u_i|,|v_i|,r_i)]R^{-1}(\psi_i) \\ &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \end{aligned} \tag{19}$$

where,

$$\begin{aligned} g_{11} =& \frac{-d_{11}}{m_{11}}\mathrm{c}^2\psi_i + (\frac{m_{11}}{m_{22}} - \frac{m_{22}}{m_{11}})r_i\mathrm{s}\psi_i\mathrm{c}\psi_i + \frac{-d_{22}}{m_{22}}\mathrm{s}^2\psi_i \\ g_{12} =& (\frac{-d_{11}}{m_{11}} - \frac{-d_{22}}{m_{22}})\mathrm{s}\psi_i\mathrm{c}\psi_i + \frac{m_{11}}{m_{22}}r_i\mathrm{s}^2\psi_i + \frac{m_{22}}{m_{11}}r_i\mathrm{c}^2\psi_i \\ & - r_i \\ g_{21} =& (\frac{-d_{11}}{m_{11}} - \frac{-d_{22}}{m_{22}})\mathrm{s}\psi_i\mathrm{c}\psi_i - \frac{m_{11}}{m_{22}}r_i\mathrm{c}^2\psi_i - \frac{m_{22}}{m_{11}}r_i\mathrm{s}^2\psi_i \\ & + r_i \\ g_{22} =& \frac{-d_{11}}{m_{11}}\mathrm{s}^2\psi_i + (\frac{m_{22}}{m_{11}} - \frac{m_{11}}{m_{22}})r_i\mathrm{s}\psi_i\mathrm{c}\psi_i + \frac{-d_{22}}{m_{22}}\mathrm{c}^2\psi_i. \end{aligned} \tag{20}$$

III. FEEDBACK DECOUPLING

We are inspired by the formation control of ODINs [18]. The matrix $\mathbf{A}([r_i],[\psi_i],[|u_i|],[|v_i|])$ is a nonlinear matrix function of $r_i, \psi_i, |u_i|, |v_i|$ for $i = 1,2,\cdots,N$. We can decompose $\mathbf{A}$ into two parts: a diagonal matrix $\mathbf{A}_\lambda$ and another matrix $\mathbf{A}_\Delta$. i.e.

$$\mathbf{A}([r_i],[\psi_i]) = \mathbf{A}_\lambda([|u_i|],[|v_i|]) + \mathbf{A}_\Delta([r_i],[\psi_i],[|u_i|],[|v_i|]) \tag{21}$$

The matrix $\mathbf{A}_\Delta$ will be viewed as a perturbation caused by lack of symmetry and by other nonlinear properties, while $\mathbf{A}_\lambda = \lambda I_{2N}$. The behaviors of the AUVs are like those of ODINs if $\mathbf{A}_\Delta$ can be tolerated. For the ith AUV, let $\lambda_i = min(-\frac{d_{11}}{m_{11}}, -\frac{d_{22}}{m_{22}})$ and we will select $\lambda = \min_i\{\lambda_i\}$. Note that even though λ is still a function of $|u_i|$ and $|v_i|$ for $i = 1,2,...,N$, the values of $\frac{d_{11}}{m_{11}}$ and $\frac{d_{22}}{m_{22}}$ are always positive, hence λ is always negative. This reflects the dissipative effect on vehicle velocity caused by the fluid. When u_i and v_i are known, the H-infinity norm of $\mathbf{A}_\Delta$ can be minimized with respect to ψ_i and r_i . Then the formation dynamic equation can be rewritten as the following:

$$\ddot{X} = \mathbf{A}_\lambda\dot{X} + \mathbf{A}_\Delta([r_i],[\psi_i])\dot{X} + \Gamma([\psi_i])U. \tag{22}$$

where we have dropped the dependence of $|u_i|$ and $|v_i|$ from $\mathbf{A}_\lambda$ and $\mathbf{A}_\Delta$ to indicate that such dependence will not affect our later results unless explicitly stated.

Because the perturbation term satisfies

$$\|\mathbf{A}_\Delta\dot{X}\|^2 = \dot{X}^T\mathbf{A}_\Delta^T\mathbf{A}_\Delta\dot{X} \le \lambda_{\max}(\mathbf{A}_\Delta^T\mathbf{A}_\Delta)\|\dot{X}\|^2, \tag{23}$$

we would like to design control laws such that the worst case perturbations from $\dot{X}$ is tolerated. In other words, we want (22) to be robust stable for all $\mathbf{A}_\Delta$ s.t.$\|\mathbf{A}_\Delta\|_\infty \le \sigma$ where $\sigma^2 = \lambda_{\max}(\mathbf{A}_\Delta^T\mathbf{A}_\Delta)$. Since σ is a function of ψ_i, and r_i. Suppose that r_i is bounded, which has practical implication since in reality, the vehicle can not be steered infinitely fast. Then the eigenvalues of $\mathbf{A}_\Delta^T\mathbf{A}_\Delta$ are bounded because only the sine and the cosine functions of ψ_i are involved in $\mathbf{A}_\Delta$, therefore $\sigma < \infty$ if the speed u_i and v_i are bounded for all i.

Suppose the value of σ has been determined, an H_∞ controller can be designed for the unperturbed system

$$\ddot{X} = \mathbf{A}_\lambda\dot{X} + \Gamma([\psi_i])U. \tag{24}$$

As indicated in [18], the system (24) possesses the property where the formation shape dynamics regarding $[\rho_j]$ and the formation center dynamics regarding q_c are decoupled. Therefore, the controller for the shape and the controller for the translation can be designed separately. In [18], we have designed a controller to achieve the following goals:

$$\rho_j \to \rho_{jd}, \dot{\rho}_j \to \dot{\rho}_{jd}, q_c \to q_{cd}, \dot{q}_c \to \dot{q}_{cd} \tag{25}$$

for $j = 1,2,\cdots,N-1$. Where ρ_{jd} is the desired value of the jth Jacobi vector and q_{cd} is the desired trajectory of formation center. As in most literatures, the formation shape is always constant, i.e. ρ_{jd} is constant and $\dot{\rho}_{jd} = 0$. Such that the formation center can track a defined trajectory and all the AUVs can keep a desired constant shape simultaneously. In the next section, we show that the gain of the linear controller can be selected to achieve robust stability for the coupled system (22).

IV. FORMATION CONTROLLER DESIGN

The full dynamics of the formation system consist of the formation dynamic equation (22) and AUV orientation equation (6), which formulates an inner-outer loop system. The vehicle orientation sub-system is in the inner loop and the AUVs formation sub-system is in the outer loop. We use feedback linearization method for the AUV orientation subsystem and then design the controller for the steering dynamics as in [18], which ensures that the steering dynamics can be controlled faster than the formation dynamics. Then we design a robust controller to stabilize the velocity subsystem of the formation which is described by equation (22) and a designed position controller for tracking by applying the Lyapunov method. The combination of the velocity and the position controller achieve robust stability of the formation dynamics under perturbations caused by lack of symmetry.

A. The Orientation Controller

For the orientation subsystem (6) of each AUV, for $i = 1,2,\cdots,N$, let the feedback linearization control be

$$\mathbf{V}_i = \frac{1}{m_{33}}\tau_{\psi_i} + \frac{(m_{11}-m_{22})}{m_{33}}u_i v_i, \tag{26}$$

The orientation equation can be rewritten as the following:

$$\begin{bmatrix} \dot{\psi}_i \\ \dot{r}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{d_{33}}{m_{33}} \end{bmatrix} \begin{bmatrix} \psi_i \\ r_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{V}_i, \tag{27}$$

Since the orientation sub-system is controllable, the linear state feedback controller for yaw moment is then

$$\mathbf{V}_i = \ddot{\psi}_{id} + \frac{d_{33}}{m_{33}}\dot{\psi}_{id} - k_1^{\psi_i}(\psi_i - \psi_{id}) - k_2^{\psi_i}(r_i - \dot{\psi}_{id}) \tag{28}$$

where $k_1^{\psi_i}, k_2^{\psi_i} > 0$ are controller gains determined by the pole assignment for $i = 1,2,\cdots,N$. This system then behaves identically to the steering control systems for the ODIN formation in [18].

B. The Robust Velocity Controller

Let $Z = \dot{X} = [\dot{\rho}_1^T, \ldots, \dot{\rho}_{N-1}^T, \dot{q}_c^T]^T$, the equation (22) can be rewritten as the following equation:

$$\dot{Z} = \mathbf{A}_\lambda Z + \mathbf{A}_\Delta([r_i],[\psi_i])Z + \Gamma([\psi_i])U. \tag{29}$$

Considering the control goal (25), we first derive the dynamic equation of the error vector $Z_e = Z - Z_d$, where $Z_d = [\dot{\rho}_{1d}, \ldots, \dot{\rho}_{(N-1)d}, \dot{q}_{cd}]^T$ is the desired velocity vector of the formation system. We have

$$\dot{Z}_e = \mathbf{A}_\lambda Z_e + \mathbf{A}_\Delta([r_i],[\psi_i])Z_e + \mu, \tag{30}$$

where,

$$\mu = \Gamma([\psi_i])U + \mathbf{A}_\lambda Z_d + \mathbf{A}_\Delta([r_i],[\psi_i])Z_d - \dot{Z}_d. \tag{31}$$

We want to emphasize here that the feedback decoupling control μ contains both the feedback decoupling control for the formation shape dynamics represented by the first $2(N-1)$ elements of μ as a vector, and the decoupling control for the formation center dynamics, represented by the last 2 elements of μ. For the formation shape control, if constant formations are desired i.e. $\dot{\rho}_{jd} = 0$ for $j = 1,2,...,N-1$, as in most literature, then the first $2(N-1)$ elements of $\mathbf{A}_\Delta([r_i],[\psi_i])Z_d$ vanish. In addition, the first $2(N-1)$ elements of $\mathbf{A}_\lambda Z_d$ and $\dot{Z}_d$ also vanish. Therefore, there is **no extra energy** spent to achieve feedback decoupling for the formation shape dynamics. For the formation center dynamics, since $\dot{q}_{cd}$ is usually not zero, extra energy in μ is required to cancel the last 2 elements of $\mathbf{A}_\Delta([r_i],[\psi_i])Z_d$. It can also be shown that $\Gamma([\psi_i])$ is always nonsingular. So that μ can always be implemented by U.

To design μ we consider (30) and define the output as the state Z_e and the perturbation as $w = \mathbf{A}_\Delta Z_e$.

Theorem 1: Suppose that the values of the angular speed r_i are bounded. Suppose $\gamma > 0$ can be determined to satisfy $\|\mathbf{A}_\Delta\|_\infty \leq \frac{1}{\gamma}$ where $\gamma \geq \frac{1}{\sqrt{\lambda^2+2}}$ and $|\lambda| \neq \frac{1}{\gamma} - \gamma$. Let

$$\alpha = \begin{cases} \dfrac{-\lambda + \sqrt{\lambda^2 - (\frac{1}{\gamma^2} - 2)}}{\frac{1}{\gamma^2} - 2} & \text{when } \gamma \in \left[\frac{1}{\sqrt{\lambda^2+2}}, \frac{1}{\sqrt{2}}\right) \\ 0 & \text{when } \gamma = \frac{1}{\sqrt{2}} \\ \dfrac{-\lambda - \sqrt{\lambda^2 - (\frac{1}{\gamma^2} - 2)}}{\frac{1}{\gamma^2} - 2} & \text{when } \gamma \in \left(\frac{1}{\sqrt{2}}, \infty\right). \end{cases} \tag{32}$$

Then the controller $\mu = -\alpha Z_e$ robustly stabilizes the formation velocity system (30), i.e.$\|T_{Z_e w}\|_\infty \leq \gamma$, where $T_{Z_e w}$ is the closed loop transfer function from w to Z_e.

Proof 1: Let $\mu = KZ_e$ in equation (30), then the closed loop system is written as the following:

$$\dot{Z}_e = (\mathbf{A}_\lambda + K)Z_e + w. \tag{33}$$

The transfer function of the closed loop system is

$$T_{Z_e w} = \left[\begin{array}{c|c} \mathbf{A}_\lambda + K & I \\ \hline I & 0 \end{array}\right]. \tag{34}$$

By the bounded real lemma [24], $\|T_{z_e w}\|_\infty \leq \gamma$ if and only if there exists a positive definite matrix solution $\mathbf{L}$ to the Algebraic Riccati Equation

$$(\mathbf{A}_\lambda + K)^*\mathbf{L} + \mathbf{L}(\mathbf{A}_\lambda + K) + \frac{1}{\gamma^2}\mathbf{L}^2 + I = 0 \tag{35}$$

and

$$\mathbf{A}_\lambda + K + \frac{1}{\gamma^2}\mathbf{L} \tag{36}$$

has no eigenvalues on the imaginary axis.

Now among all possible K, let us select $K = -\mathbf{L}$. The Algebraic Riccati Equation (35) is rewritten as:

$$\mathbf{A}_\lambda^*\mathbf{L} + \mathbf{L}\mathbf{A}_\lambda + (\frac{1}{\gamma^2} - 2)\mathbf{L}^2 + I = 0. \tag{37}$$

We must show that there exists a stabilizing solution of equation (37). And in addition, this solution makes $\mathbf{A}_\lambda - \mathbf{L}$ Hurwitz and the matrix $\mathbf{A}_\lambda + (\frac{1}{\gamma^2} - 1)L$ non-singular. If this is true, then the state feedback controller $\mu = -\mathbf{L}Z_e$ robustly stabilizes the system.

Because $\mathbf{A}_\lambda$ is a diagonal matrix, we can find $\mathbf{L}$ from (37) as a diagonal matrix αI where $\alpha = \frac{-\lambda \pm \sqrt{\lambda^2 - (\frac{1}{\gamma^2}-2)}}{\frac{1}{\gamma^2}-2}$. We first observe that γ must satisfy $(\frac{1}{\gamma^2}-2) \leq \lambda^2$. Hence $\gamma \geq \frac{1}{\sqrt{\lambda^2+2}}$, so that a real solution for α exists.

Next, the solution α must be positive. When $\frac{1}{\gamma^2} < 2$ e.g. $\gamma > \frac{1}{\sqrt{2}}$, the solution can only be $\alpha = \frac{-\lambda - \sqrt{\lambda^2 - (\frac{1}{\gamma^2}-2)}}{\frac{1}{\gamma^2}-2}$. In addition, this solution satisfies $\lambda - \alpha < 0$ and $\lambda + (\frac{1}{\gamma^2} - 1)\alpha \neq 0$ for λ and γ if $|\lambda| \neq \frac{1}{\gamma} - \gamma$. When $\frac{1}{\gamma^2} = 2$, the solution can only be $\alpha = 0$, which is also a satisfying solution.

When $\frac{1}{\gamma^2} > 2$, the solution $\alpha = \frac{-\lambda + \sqrt{\lambda^2 - (\frac{1}{\gamma^2}-2)}}{\frac{1}{\gamma^2}-2}$ satisfies all conditions. We also notice that the solution $\alpha = \frac{-\lambda - \sqrt{\lambda^2 - (\frac{1}{\gamma^2}-2)}}{\frac{1}{\gamma^2}-2}$ satisfies all condition but requires $|\lambda| \neq \frac{1}{\gamma} - \gamma$. Therefore, we select $\alpha = \frac{-\lambda + \sqrt{\lambda^2 - (\frac{1}{\gamma^2}-2)}}{\frac{1}{\gamma^2}-2}$.

According to the Small Gain Theorem [24], the system (33) with perturbation $\mathbf{A}_\Delta$ is well-posed and internally stable for all $\mathbf{A}_\Delta \in RH_\infty$ with $\|\mathbf{A}_\Delta\|_\infty \leq 1/\gamma$ if and only if $\|T_{z_e w}\|_\infty < \gamma$. ■

C. The Position Controller

Now that the velocity sub-system with perturbation is **robustly stable** under the control $\mu = -\mathbf{L}Z_e$. We add $-K_1 X_e$ where $X_e = X - X_d$ into the control law μ so that $\mu' = -\mathbf{L}Z_e - K_1 X_e$ to achieve position control for the following full system.

$$\begin{cases} \dot{X}_e = Z_e \\ \dot{Z}_e = \mathbf{A}_\lambda Z_e + \mathbf{A}_\Delta([r_i],[\psi_i])Z_e + \mu'. \end{cases} \tag{38}$$

Theorem 2: Consider the control law

$$\mu' = -\mathbf{L}Z_e - K_1 X_e \tag{39}$$

where, $\mathbf{L}$ is the unique stabilizing solution of the Algebraic Riccati Equation (37) and K_1 is a positive definite matrix. Then as $t \to \infty$, we have that $X_e \to 0$ and $Z_e \to 0$ for the system (38).

Proof 2: The unperturbed formation system can be rewritten as the following:

$$\begin{cases} \dot{X}_e = Z_e \\ \dot{Z}_e = \mathbf{A}_\lambda Z_e + \mu'. \end{cases} \tag{40}$$

Even though this is not a linear system due to the fact that $\mathbf{A}_\lambda$ depends on $|u_i|$ and $|v_i|$ for $i = 1,2,...,N$, its behavior is very close to a linear system, since λ is always negative. It is not difficult to show that as long as K_1 is a positive definite matrix, then the closed loop system (38) is asymptotically stable. Furthermore, system (38) is robustly stable under the perturbation $\mathbf{A}_\Delta$. ■

Use μ' for the formation system to calculate the surge and sway control forces for all the AUVs by solving the equation (31). We have the following algebraic equations

$$U = \Gamma(\psi_i)^{-1}(\mu' - \mathbf{A}_\lambda Z_d - \mathbf{A}_\Delta(r_i,\psi_i)Z_d + \dot{Z}_d). \tag{41}$$

Where the inverse matrix $\Gamma(\psi_i)^{-1}$ exists according to its definition (17) because the matrix Φ and $H(\psi)$ are nonsingular.

We summarize the controller design procedure as an algorithm below:

Step1: Calculate the value of r_i and ψ_i $(i = 1,2,\cdots,N)$ for all the AUVs according to the equation(28) and (6).
Step2: Compute $\sigma = \|\mathbf{A}_\Delta\|_\infty$ with the known values of r_i,ψ_i, and λ according to the hydrodynamic parameters of the AUV;
Step3: Let $\gamma = \frac{1}{\sigma}$. If $\gamma \geq \frac{1}{\sqrt{\lambda^2+2}}$, then go to the next step. If not, break out;
Step4: Compute the value for α and get the gain matrix $\mathbf{L} = \alpha I$ and choose K_1 as a positive definite matrix. Compute the control $\mu' = -\mathbf{L}Z_e - K_1 X_e$;
Step5: Solve equation(41) to get the control force U for the AUVs.

V. SIMULATION RESULTS

In this section, we carry out simulation to demonstrate the effectiveness of proposed formation controllers. The model parameters are adapted as follows [12] [25]: $m_{11} = 200kg, m_{22} = 250kg, m_{33} = 80kg$, $d_{11} = (70+100|u|)kg/s$, $d_{22} = (100+200|v|)kg/s$, $d_{33} = (50+100|r|)kg/s$. There are three vehicles which are initialized as follows: $(x_1,y_1) = (25\text{m},5\text{m})$, $(x_2,y_2) = (-5\text{m},10\text{m})$, $(x_3,y_3) = (10\text{m},8\text{m})$, $u_1 = v_1 = u_2 = v_2 = u_3 = v_3 = 1\text{m/s}$, $\psi_1 = 0.1\text{rad}$, $\psi_2 = 0.4\text{rad}$, $\psi_3 = 0.7\text{rad}$, $r_1 = r_2 = 0.1\text{rad/s}$. The Jacobi vectors are defined the same as in [18]. Let $\rho_{1d} = (0,10)$ and $\rho_{2d} = (20,0)$, the desired formation shape is a isosceles triangle. The desired trajectory is an sinusoidal line taken as $q_{cd} = [t,\ 30 * sin(0.1t)]^T$. Figure 1 shows the trajectories of the three AUVs. The positions are marked by '▶' every 50 seconds. From the Figure 1 we can see that the three AUVs form the triangular formation immediately and keep moving in fixed formation. The formation center tracks the sinusoidal trajectory well. Figure 2 shows that the surge velocities converge to 1m/s and the sway velocities converge to 0 and the yaw angle velocities converge to 0. Another simulation about six AUVs controlled to keep a polygon formation is plotted in Figure 3.

VI. CONCLUSION

This paper presents a novel approach of formation control for AUVs based on the Jacobi shape theory. It is shown that this approach reduces the complexity of formation controller design. Future work will include collision avoidance, obstacle avoidance and 3D trajectory tracking.

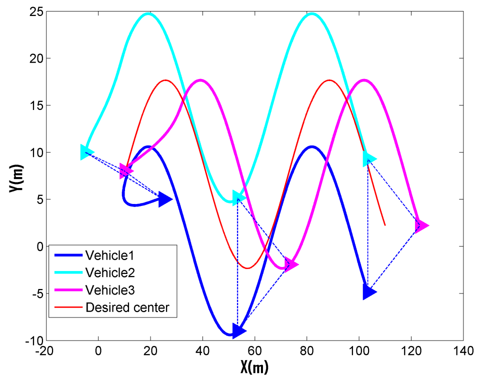

Fig. 1. Three AUVs forms a triangular formation moving on a sinusoidal line.

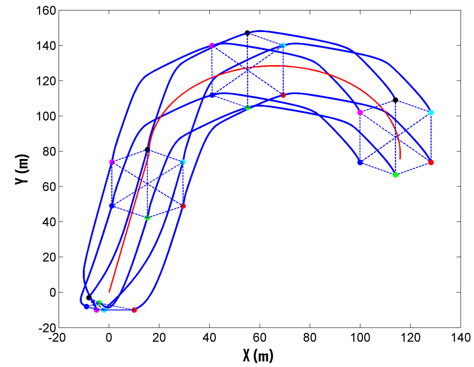

Fig. 3. Six AUVs forms a polygon formation moving on a desired trajectory.

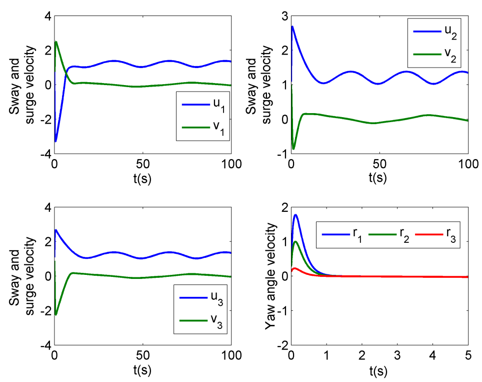

Fig. 2. Yaw angle velocity matching of the AUVs.

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