Exploring Scalar Fields Using Multiple Sensor Platforms: Tracking Level Curves

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***Abstract*—Autonomous mobile sensor networks are employed to measure large scale environmental scalar fields. Yet an optimal strategy for mission design addressing both the cooperative motion control and the collaborative sensing is still under investigation. We develop one strategy which uses four moving sensor platforms to explore a noisy scalar field defined in the plane; each platform can only take one measurement at a time. We derive a Kalman filter in conjunction with a nonlinear filter to produce estimates for the field value, the gradient and the Hessian along the averaged trajectories of the moving platforms. The shape of the platform formation is designed to minimize error in the estimates, and a cooperative control law is designed to asymptotically achieve the optimal formation. We develop a motion control law to allow the center of the platform formation to move along level curves of the averaged field. Convergence of the control laws are proved, and performance of both the filters and the control laws are demonstrated in simulated ocean fields.**

I. INTRODUCTION

The mission of measuring a scalar field, such as a temperature or salinity field, is encountered in ocean science and meteorology. Since the scalar field is often distributed across a large area, it would take too many sensors to obtain a snapshot of the field if the sensors are installed at fixed locations. Mobile sensor networks are ideal candidates for such missions: a small number of moving sensor platforms can patrol a large area, taking measurements along their motion trajectories.

Mission design for a mobile sensor network requires combination of cooperative control and collaborative sensing. This is because the quality of collected information is coupled with the motion of sensor platforms. Recent theoretical and experimental developments suggest that a balance between data collection and feasible motion is key to mission success [1], [2]. Finding an optimal strategy is a challenging task.

In this paper, we design a mission for a mobile sensor network to track level curves of a noisy scalar field. We study scalar fields defined in the plane with spatially correlated noise. Combining the sensor measurements from different platforms along their trajectories, we develop a Kalman filter and a nonlinear filter that provide estimates for the scalar field, its gradient, and its Hessian. A cooperative control law then uses these estimates to control the center of the mobile sensor network to move along a level curve of the scalar field.

Related to our work, mobile sensor networks can also be applied to climb gradients of a scalar field [3], to monitor environmental boundaries such as oil spills or chemical plumes in the ocean [4]–[7], to patrol the perimeter of an object or a region [8]–[12], or to provide coverage over a large area [2], [13].

II. PARTICLE MOTION IN A SCALAR FIELD

Suppose a Newtonian particle with unit mass is free to move in the plane with its position represented by $\mathbf{r}_c$. The system equation for such particle is $\ddot{\mathbf{r}}_c = \mathbf{f}_c$ where $\mathbf{f}_c$ represents the total force on this particle. This Newton's equation can be written in an equivalent Frenet-Serret form which is more convenient for tracking purposes [2], [13].

We define a unit velocity vector $\mathbf{x}_2$ as $\mathbf{x}_2 = \frac{\dot{\mathbf{r}}_c}{\alpha}$ where $\alpha = \|\dot{\mathbf{r}}_c\|$, and define a unit vector $\mathbf{y}_2$ as the vector perpendicular to $\mathbf{x}_2$ but forming a right handed frame with $\mathbf{x}_2$ so that $\mathbf{x}_2$ and $\mathbf{y}_2$ lie in the plane of the page and the vector $\mathbf{x}_2 \times \mathbf{y}_2$ points towards the reader. Then the steering control can be defined as $u_c = \frac{1}{\alpha^2}\mathbf{f_c}\cdot\mathbf{y}_2$ and the speed control can be defined as $v_c = \mathbf{f_c}\cdot\mathbf{x}_2$. We have the following equations:

$$\begin{aligned}\dot{\mathbf{x}}_2 &= u_c\alpha\mathbf{y}_2\\ \dot{\mathbf{y}}_2 &= -u_c\alpha\mathbf{x}_2\,.\end{aligned} \tag{1}$$

The equation for speed control is

$$\dot{\alpha} = v_c\,. \tag{2}$$

The equations (1) and (2) describe the particle motion in the Frenet-Serret form.

We let $v_c = -k_1(\alpha - 1)$. As time $t \to \infty$, α converges to unit speed exponentially with a rate determined by $k_1 > 0$.

Let $z(\mathbf{r})$ be an unknown smooth scalar function in the plane. With the speed of the particle under control, we design a steering control u_c for the particle so that it will track a level curve of $z(\cdot)$. The procedure can be found in our previous works [2], [13]. Here we briefly summarize and explain the results.

At any time instant t, there is a level curve of $z(\cdot)$ passing through $\mathbf{r}_c$. At this position $\mathbf{r}_c$, we let $\mathbf{y}_1$ be the unit vector in the direction of the gradient of the field $z(\cdot)$, and let $\mathbf{x}_1$ be the unit tangent vector to the level curve. By convention, $\mathbf{x}_1$ and $\mathbf{y}_1$ form a right handed coordinate frame with $\mathbf{x}_1 \times \mathbf{y}_1$ pointing to the reader.

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For convenience, we introduce a variable $\theta \in (-\pi, \pi]$ such that

$$\cos\theta = \mathbf{x}_1 \cdot \mathbf{x}_2$$
$$\sin\theta = -\mathbf{y}_1 \cdot \mathbf{x}_2 \ . \qquad (3)$$

Along the trajectory of the center, it can be shown that

$$\dot{\theta} = \alpha(\kappa_1 \cos\theta + \kappa_2 \sin\theta - u_c) \qquad (4)$$

where

$$\kappa_1 = -\frac{\mathbf{x}_1^T \nabla^2 z \mathbf{x}_1}{\|\nabla z\|} \text{ and } \kappa_2 = \frac{\mathbf{x}_1^T \nabla^2 z \mathbf{y}_1}{\|\nabla z\|}, \qquad (5)$$

and $\nabla^2 z$ represents the Hessian of the scalar field $z(\cdot)$. Meanwhile, along the trajectory of the center, the value of z satisfies

$$\dot{z} = -\alpha \|\nabla z\| \sin\theta \ . \qquad (6)$$

Our goal is to design the tracking control u_c so that $\theta \to 0$ and $z \to C$ asymptotically where C is a given constant.

Let $\widetilde{f}(z)$ be the derivative function of a function $\hbar(z)$ that satisfies certain technical conditions as in [14]. We design the control law to be

$$u_{\mathrm{c}} = \kappa_1 \cos\theta + \kappa_2 \sin\theta - 2\widetilde{f}(z)\|\nabla z\| \cos^2(\frac{\theta}{2}) + \sin(\frac{\theta}{2}) \ . \qquad (7)$$

The proof of the convergence of this steering control law is similar to those in our previous works [14], [13], and [2].

III. Kalman Filter Design

In the steering control law (7), we observe that the field value z, the gradient ∇z, and the curvatures κ_1 and κ_2—which depend on the Hessian $\nabla^2 z$—are required by the particle to track a level curve of the scalar field $z(\cdot)$. In most practical situations, since the field is noisy and the sensing devices are imperfect, it is difficult to estimate the field value, the gradient, and the Hessian using a single sensor platform. The key idea here is to employ multiple moving sensor platforms to obtain the necessary estimates cooperatively to reduce noise. This requires the platforms to be in a formation. The center of the formation as well as each moving platform are modeled as Newtonian particles. The center will be controlled to travel at unit speed and be steered to follow a level curve using the steering control law (7).

Let the positions of the sensor platforms at time t be $\mathbf{r}_i(t) \in \mathbb{R}^2$ where $i = 1, 2, ..., N$. Let $\mathbf{r}_{\mathrm{c}}$ be the center of the platform formation i.e. $\mathbf{r}_{\mathrm{c}}(t) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i(t)$. We assume that the measurement taken by the ith platform is modeled as

$$y_i(\mathbf{r}_i) = z(\mathbf{r}_i) + w(\mathbf{r}_i) + n_i \qquad (8)$$

where $n_i \sim \mathcal{N}(0, \sigma_i^2)$ are i.i.d. Gaussian noise and $w(\mathbf{r}_i)$ are spatially correlated Gaussian noise. We define the following $N \times 1$ vectors:

$$\mathbf{y} = [y_i], \ \mathbf{z} = [z(\mathbf{r}_i)], \ \mathbf{n} = [n_i], \ \mathbf{w} = [w(\mathbf{r}_i)], \qquad (9)$$

and assume that $\mathbf{n}$ and $\mathbf{w}$ are stationary, i.e., their statistics are time invariant. These assumptions are idealizations for physical scalar fields in the ocean or atmosphere.

Let the moment when new measurements are available be t_k where k is an integer index. To simplify the derivation, we will not consider the asynchronicity in the measurements; we assume that all platforms will have new measurements at time t_k.

Let $\mathbf{r}_{i,k} = \mathbf{r}_i(t_k)$ and $\mathbf{r}_{\mathrm{c},k} = \mathbf{r}_{\mathrm{c}}(t_k)$. The function $z(\mathbf{r}_{i,k})$ can be locally approximated by a Taylor series. If $\mathbf{r}_{i,k}$ is close to $\mathbf{r}_{\mathrm{c},k}$, then it is sufficient to use the Taylor series up to second order. Let $z_{i,k} = z(\mathbf{r}_{i,k})$, then

$$z_{i,k} = z(\mathbf{r}_{\mathrm{c},k}) + (\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k})^T \nabla z(\mathbf{r}_{\mathrm{c},k}) + \frac{1}{2}(\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k})^T \nabla^2 z(\mathbf{r}_{\mathrm{c},k})(\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k}) \qquad (10)$$

for $i = 1, 2, ..., N$. We are interested in estimates of $z(\mathbf{r}_{\mathrm{c},k})$, $\nabla z(\mathbf{r}_{\mathrm{c},k})$, and $\nabla^2 z(\mathbf{r}_{\mathrm{c},k})$. Other than providing insights on the structure of the scalar field, these estimates are also used in the steering control for the center of the formation.

In order to solve for the field value, the gradient, and the Hessian—altogether six unknowns, we must let $N \geq 6$ if the sensor platforms are not moving. We are able to reduce the number of sensor platforms by utilizing cooperatively controlled motion of the platforms. The intuition is that measurements from different time instances can be combined in the estimation process—a key idea for filter designs. In this paper we show the case $N = 4$. We develop discrete filters to find an estimate $z_{\mathrm{c},k}$ for $z(\mathbf{r}_{\mathrm{c},k})$, an estimate $\mathbf{d}_{\mathrm{c},k}$ for $\nabla z(\mathbf{r}_{\mathrm{c},k})$, and an estimate $H_{\mathrm{c},k}$ for the Hessian $\nabla^2 z(\mathbf{r}_{\mathrm{c},k})$.

Let $\mathbf{s}_k = [z(\mathbf{r}_{\mathrm{c},k}), \nabla z(\mathbf{r}_{\mathrm{c},k})^T]^T$. Let C_k be the $N \times 3$ matrix where its ith row vector is defined by $(C_k)_i = \begin{bmatrix} 1 & (\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k})^T \end{bmatrix}$ for $i = 1, 2, ..., N$. Let D_k be the $N \times 4$ matrix with its ith row vector defined by $\frac{1}{2}((\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k}) \otimes (\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k}))^T$ where $\otimes$ is the Kronecker product. For any 2×2 matrix H, we use the notation $\vec{H}$ to represent a column vector defined by rearranging the elements of H as follows

$$\vec{H} = [H_{11}, H_{21}, H_{12}, H_{22}]^T. \qquad (11)$$

Then the Taylor expansions for all sensor platforms near $\mathbf{r}_{\mathrm{c},k}$ can be re-written in a vector form as

$$\mathbf{z}_k = C_k \mathbf{s}_k + D_k \vec{\nabla}^2 z(\mathbf{r}_{\mathrm{c},k}) \qquad (12)$$

where $\mathbf{z}_k = \mathbf{z}(t_k)$, and $\vec{\nabla}^2 z(\mathbf{r}_{\mathrm{c},k})$ is a 4×1 column vector obtained by rearranging elements of the Hessian $\nabla^2 z(\mathbf{r}_{\mathrm{c}})$ as defined by (11).

Let $\widetilde{C}_k$ be the $N \times 3$ matrix with its ith row defined by $(\widetilde{C}_k)_i = \begin{bmatrix} 1 & (\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k-1})^T \end{bmatrix}$ for $i = 1, 2, ..., N$. Let $\widetilde{D}_k$ be the $N \times 4$ matrix with its ith row vector defined as $\frac{1}{2}((\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k-1}) \otimes (\mathbf{r}_{i,k} - \mathbf{r}_{\mathrm{c},k-1}))^T$. Then the Taylor expansion near $\mathbf{r}_{\mathrm{c},k-1}$ is

$$\mathbf{z}_k = \widetilde{C}_k \mathbf{s}_{k-1} + \widetilde{D}_k \vec{\nabla}^2 z(\mathbf{r}_{\mathrm{c},k-1}). \qquad (13)$$

Equating (12) and (13), we can solve for the relationship between $\mathbf{s}_k$ and $\mathbf{s}_{k-1}$ as

$$\mathbf{s}_k = A_{k-1}^{\mathrm{s}} \mathbf{s}_{k-1} - (C_k^T C_k)^{-1} C_k^T (D_k \vec{\nabla}^2 z(\mathbf{r}_{\mathrm{c},k}) - \widetilde{D}_k \vec{\nabla}^2 z(\mathbf{r}_{\mathrm{c},k-1})) \qquad (14)$$

where

$$A_{k-1}^{\mathrm{s}} = (C_k^T C_k)^{-1}(C_k^T \widetilde{C}_k). \qquad (15)$$

We formulate the problem into the framework of Kalman filters. Suppose that $\vec{H}_{c,k}$ is an estimate for the Hessian $\vec{\nabla}^2 z(\mathbf{r}_{c,k})$ in vector form. Let

$$\mathbf{h}_{k-1} = -(C_k^T C_k)^{-1} C_k^T (D_k \vec{H}_{c,k} - \widetilde{D}_k \vec{H}_{c,k-1}). \quad (16)$$

We then rewrite (14) as

$$\mathbf{s}_k = A_{k-1}^{s} \mathbf{s}_{k-1} + \mathbf{h}_{k-1} + \boldsymbol{\epsilon}_{k-1} \quad (17)$$

where we have introduced the $N \times 1$ noise vector $\boldsymbol{\epsilon}_{k-1}$ which accounts for positioning errors, estimation errors for the Hessians, and errors caused by higher order terms omitted from the Taylor expansion. We assume that $\boldsymbol{\epsilon}_{k-1}$ are i.i.d Gaussian with zero mean and covariance matrix M_{k-1}.

Equation (8) can also be written in vector form as

$$\mathbf{y}_k = C_k \mathbf{s}_k + D_k \vec{H}_{c,k} + \mathbf{w}_k + D_k \mathbf{e}_k + \mathbf{n}_k \quad (18)$$

where $\mathbf{e}_k$ represents the error in the estimate of the Hessian. Let $W_k = E[\mathbf{w}_k \mathbf{w}_k^T]$, $U_k = E[\mathbf{e}_k \mathbf{e}_k^T]$, and $R_k = E[\mathbf{n}_k \mathbf{n}_k^T]$. The noise $\mathbf{w}_k$ is "colored" because it models the spatial correlation of the field. Let $E[\mathbf{w}_k \mathbf{w}_{k-1}^T] = V_k$. We suppose that W_k, R_k and V_k are known once the positions of the platforms are known. This assumption is reasonable since the statistical property of ocean field and atmospheric field are usually known from accumulated observational data over a long period of time.

Instead of using (18) directly as the system output equation, we define $F_k = (C_k^T C_k)^{-1} C_k^T$ and let $\widetilde{\mathbf{y}}_k = F_k \mathbf{y}_k$ so that

$$\widetilde{\mathbf{y}}_k = \mathbf{s}_k + F_k \mathbf{w}_k + F_k (D_k \vec{H}_{c,k} + D_k \mathbf{e}_k + \mathbf{n}_k). \quad (19)$$

This $\widetilde{\mathbf{y}}_k$ is a combination of the platform measurements $\mathbf{y}_k$ and has the same dimension as $\mathbf{s}_k$.

In order to design a Kalman filter with colored noise $\mathbf{w}_k$, we model $\mathbf{w}_k$ as

$$\mathbf{w}_k = A_{k-1}^{w} \mathbf{w}_{k-1} + \boldsymbol{\eta}_{k-1} \quad (20)$$

where $\boldsymbol{\eta}_{k-1}$ is white noise with correlation matrix $Q_k = E[\boldsymbol{\eta}_k \boldsymbol{\eta}_k^T]$. Then, because

$$\begin{aligned} V_k &= E[\mathbf{w}_k \mathbf{w}_{k-1}^T] = A_{k-1}^{w} E[\mathbf{w}_{k-1} \mathbf{w}_{k-1}^T] = A_{k-1}^{w} W_{k-1} \\ W_k &= E[\mathbf{w}_k \mathbf{w}_k^T] = A_{k-1}^{w} W_{k-1} (A_{k-1}^{w})^T + Q_{k-1}, \end{aligned} \quad (21)$$

we have

$$\begin{aligned} A_{k-1}^{w} &= V_k W_{k-1}^{-1} \\ Q_{k-1} &= W_k - A_{k-1}^{w} W_{k-1} (A_{k-1}^{w})^T. \end{aligned} \quad (22)$$

The equations (17), (20), and (19) are the system and output equations for the mobile sensor network with N platforms. The states are $[\mathbf{s}_k^T, \mathbf{w}_k^T]^T$, the output is $\widetilde{\mathbf{y}}_k$, the state noise are $[\boldsymbol{\epsilon}_k^T, \boldsymbol{\eta}_k^T]^T$, and the observation noise are $D_k \mathbf{e}_k + \mathbf{n}_k$. The equations for Kalman filters are obtained by canonical procedures. We may derive formulas for the Kalman filter following standard textbooks such as [15]. Here we omit this procedure due to space limit. The rest of the paper does not depend on specific forms for the Kalman filter.

IV. Estimating the Hessian

An estimate of the Hessian, $H_{c,k}$, is needed to enable the Kalman filter. At the end of the $(k-1)$th time step, we have obtained an estimate $\mathbf{s}_{k-1}(+)$ from the Kalman filter. This includes an estimate $z_{c,k-1}$ for $z(\mathbf{r}_{c,k-1})$ and an estimate $\mathbf{d}_{c,k-1}$ for $\nabla z(\mathbf{r}_{c,k-1})$. We outline the procedure to compute $H_{c,k}$ as follows:

1) Start with an estimate or an initial guess $H_{c,k-1}$.
2) Use a one-step filter to reduce noise in the new measurements.
3) Determine a level curve passing through the center of the platform formation and estimate its curvature.
4) Compute $H_{c,k}$.

A. Curvature and Hessian

The level curve passing through the center of the formation $\mathbf{r}_c$ can be parametrized by its arc-length s, hence $z(\mathbf{r}(s))$ is constant for all values of s. Suppose the gradient ∇z does not vanish along the curve. We recall that the unit normal vector to the level curve is defined as $\mathbf{y}_1(s) = \frac{\nabla z(\mathbf{r}(s))}{\|\nabla z(\mathbf{r}(s))\|}$, and at any given point, the unit tangent vector to the curve, denoted by $\mathbf{x}_1(s)$, satisfies $\mathbf{x}_1(s) \cdot \mathbf{y}_1(s) = 0$. Then we have the following Frenet-Serret equations [16]:

$$\begin{aligned} \frac{d\mathbf{x}_1(s)}{ds} &= \kappa(s)\mathbf{y}_1(s) \\ \frac{d\mathbf{y}_1(s)}{ds} &= -\kappa(s)\mathbf{x}_1(s)\,, \end{aligned} \quad (23)$$

where $\kappa(s)$ is defined as the Frenet-Serret curvature of the level curve.

With this configuration, because $\nabla z(\mathbf{r}_c) \cdot \mathbf{x}_1 = 0$ along the level curve, the derivative with respect to s is

$$\frac{d}{ds}\nabla z(\mathbf{r}_c) \cdot \mathbf{x}_1 + \nabla z(\mathbf{r}_c) \cdot \frac{d\mathbf{x}_1}{ds} = 0 \quad (24)$$

which implies

$$\mathbf{x}_1^T \nabla^2 z(\mathbf{r}_c) \mathbf{x}_1 + \|\nabla z(\mathbf{r}_c)\| \mathbf{y}_1 \cdot \kappa(s) \mathbf{y}_1 = 0 \quad (25)$$

where $\nabla^2 z(\mathbf{r}_c)$ is the Hessian of z at $\mathbf{r}_c$. Because $\mathbf{x}_1$ is the unit vector along the $\mathbf{x}_1$-axis, in the Frenet-Serret frame we have

$$\partial_{xx} z(\mathbf{r}_c) + \|\nabla z(\mathbf{r}_c)\| \kappa(s) = 0. \quad (26)$$

This suggests that we can obtain $H_{xx,k}$, the estimate for $\partial_{xx} z(\mathbf{r}_c)$, by

$$H_{xx,k} = -\left\| \mathbf{d}_{c,k} \right\| \kappa_{c,k} \quad (27)$$

where $\mathbf{d}_{c,k}$ is the estimate for the gradient $\nabla z(\mathbf{r}_{c,k})$ and $\kappa_{c,k}$ is the estimate for the curvature $\kappa(\mathbf{r}_{c,k})$.

On the other hand, we have $\nabla z(\mathbf{r}_c) \cdot \mathbf{y}_1 = \|\nabla z(\mathbf{r}_c)\|$. Taking derivatives on both sides of this equation with respect to s, we get

$$\mathbf{x}_1^T \nabla^2 z(\mathbf{r}_c) \mathbf{y}_1 - \|\nabla z(\mathbf{r}_c)\| \mathbf{y}_1 \cdot \kappa(s) \mathbf{x}_1 = \frac{d}{ds} \|\nabla z_c\| \,. \quad (28)$$

This implies that $\partial_{\rm xy} z(\mathbf{r}_{\rm c}) = \frac{d}{ds} \| \nabla z_{\rm c} \|$. Therefore, the estimate $H_{{\rm xy},k}$ for $\partial_{\rm xy} z(\mathbf{r}_{{\rm c},k})$ is

$$H_{{\rm xy},k} = \frac{d}{ds} \left\| \mathbf{d}_{{\rm c},k} \right\|. \tag{29}$$

The estimates $H_{{\rm xx},k}$ and $H_{{\rm xy},k}$ are elements of $H_{{\rm c},k}$ in the Frenet-Serret coordinate system. Since the field $z(\cdot)$ is smooth, we require $H_{{\rm yx},k} = H_{{\rm xy},k}$. We also need to find out $H_{{\rm yy},k}$ to determine $H_{{\rm c},k}$.

B. A One-step Filter to Reduce Noise

Using the known estimates at time $k-1$, we can make predictions for the field value at the positions of the sensor platforms at the kth step when the platforms will move from $\mathbf{r}_{i,k-1}$ to $\mathbf{r}_{i,k}$ as

$$\mathbf{z}_k^{\rm P} = \widetilde{C}_k \mathbf{s}_{k-1}(+) + \widetilde{D}_k \vec{H}_{{\rm c},k-1}. \tag{30}$$

The error of the prediction $\mathbf{z}_k^{\rm P}$ compared to the true value $\mathbf{z}_k$ is Gaussian i.e. $\mathbf{z}_k^{\rm P} = \mathbf{z}_k + \psi_k$. From properties of the Kalman filter, the covariance of $\mathbf{z}_k^{\rm P}$ is $G_k = E[\psi_k \psi_k^T] = \widetilde{C}_k P_{k-1}^{\rm s}(+) \widetilde{C}_k^T$ where $P_{k-1}^{\rm s}(+)$ is the error covariance in the estimate $\mathbf{s}_{k-1}(+)$.

Using the estimate of Hessian $H_{{\rm c},k-1}$ from the previous step, we also make a prediction $H_{{\rm c},k}^{\rm P}$ that satisfies $H_{{\rm c},k}^{\rm P} = H_{{\rm c},k-1}$. Using the Kalman filter, we may obtain a prediction for the $\mathbf{s}_k$ as

$$\mathbf{s}_k^{\rm P} = A_{k-1}^{\rm s} \mathbf{s}_{k-1}(+) + \mathbf{h}_{k-1}^{\rm P} \tag{31}$$

where

$$\mathbf{h}_{k-1}^{\rm P} = -(C_k^T C_k)^{-1} C_k^T (D_k \vec{H}_{{\rm c},k}^{\rm P} - \widetilde{D}_k \vec{H}_{{\rm c},k-1}). \tag{32}$$

Note the difference between $\mathbf{h}_{k-1}^{\rm P}$ and $\mathbf{h}_{k-1}$ in (16).

We then take new measurements at the kth step using all four platforms. Let $\mathbf{y}_k$ be the vector of the measurements and $\mathbf{z}_k^{\rm P}$ be the vector of the predictions. Let the updated measurements $\widehat{\mathbf{z}}_k$ be

$$\widehat{\mathbf{z}}_k = (I + G_k (W_k + R_k)^{-1})^{-1} \mathbf{z}_k^{\rm P} + (I + (R_k + W_k) G_k^{-1})^{-1} \mathbf{y}_k. \tag{33}$$

Such $\widehat{\mathbf{z}}_k$ minimizes the cost function

$$J_k = \frac{1}{2} \left[(\widehat{\mathbf{z}}_k - \mathbf{z}_k^{\rm P})^T G_k^{-1} (\widehat{\mathbf{z}}_k - \mathbf{z}_k^{\rm P}) + (\mathbf{y}_k - \widehat{\mathbf{z}}_k)(W_k + R_k)^{-1} (\mathbf{y}_k - \widehat{\mathbf{z}}_k) \right]. \tag{34}$$

As we can see, G_k serves as the weighting matrix that balances using the information from previous estimates and from current measurements.

Claim 4.1: The estimator given in equation (33) is unbiased with the error covariance matrix $(I + G_k(W_k + R_k)^{-1})^{-1} G_k$.

Proof: For simplicity we drop the subscripts k in W_k, R_k and G_k. Because $\mathbf{z}_k^P = \mathbf{z}_k + \psi_k$ and $\mathbf{y}_k = \mathbf{z}_k + \mathbf{w}_k + \mathbf{n}_k$, we have

$$\begin{aligned} \widehat{\mathbf{z}}_k &= (I + G(W+R)^{-1})^{-1} \mathbf{z}_k^{\rm P} + (I + (R+W)G^{-1})^{-1} \mathbf{y}_k \\ &= (I + G(W+R)^{-1})^{-1} (\mathbf{z}_k + \psi_k) \\ &+ (I + G(W+R)^{-1})^{-1} G(W+R)^{-1} (\mathbf{z}_k + \mathbf{w}_k + \mathbf{n}_k) \\ &= \mathbf{z}_k + (I + G(W+R)^{-1})^{-1} (\psi_k \\ &+ G(W+R)^{-1} (\mathbf{w}_k + \mathbf{n}_k)) . \end{aligned} \tag{35}$$

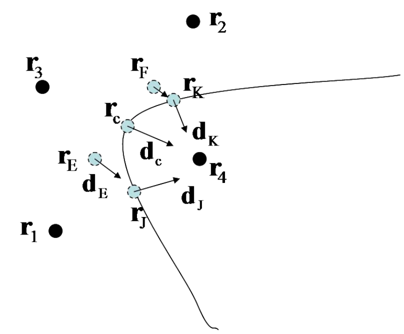

Fig. 1. *Detection of a level curve using four sensor platforms.* $\mathbf{r}_{\rm c}$ *denotes the center of the entire formation.* $\mathbf{r}_{\rm E}$ *denotes the center of the formation formed by* $\mathbf{r}_1$, $\mathbf{r}_3$ *and* $\mathbf{r}_4$. $\mathbf{r}_{\rm F}$ *denotes the center of the formation formed by* $\mathbf{r}_2$, $\mathbf{r}_3$ *and* $\mathbf{r}_4$. $\mathbf{r}_{\rm J}$ *and* $\mathbf{r}_{\rm K}$ *are located on the same level curve with* $\mathbf{r}_{\rm c}$.

Therefore $E(\widehat{\mathbf{z}}_k) = E(\mathbf{z}_k)$ because ψ_k, $\mathbf{w}_k$ and $\mathbf{n}_k$ have zero mean. The error covariance can be directly computed to be $(I + G(W+R)^{-1})^{-1} G$.■

C. Estimation of Curvature

We are ready to estimate the curvature of the level curve passing through $\mathbf{r}_{{\rm c},k}$. Since the procedure only involves information for step k, we drop the subscript k in this section for simplicity.

With a formation of four moving sensor platforms, we are able to estimate $\kappa(s)$ for the level curve at the center of the formation by the following steps:

1) Compute an estimate of the field value and gradient at the center $\mathbf{r}_{\rm c}$ using (31).
2) Considering the formation formed by $\mathbf{r}_1$, $\mathbf{r}_3$ and $\mathbf{r}_4$, obtain the estimates $z_{\rm E}$ and $\mathbf{d}_{\rm E}$ at the center $\mathbf{r}_{\rm E}$ of this three platform formation (Fig. 1) by solving the following equations for $i = 1,3,4$:

$$\widehat{z}_i = z_{\rm E} + \mathbf{d}_{\rm E} \cdot (\mathbf{r}_i - \mathbf{r}_{\rm E}) + \frac{1}{2} (\mathbf{r}_i - \mathbf{r}_{\rm E})^T H^{\rm P} (\mathbf{r}_i - \mathbf{r}_{\rm E}). \tag{36}$$

Let $\widehat{\mathbf{z}}_{\rm E} = [\widehat{z}_1, \widehat{z}_3, \widehat{z}_4]^T$, $\mathbf{s}_{\rm E} = [z_{\rm E}, \mathbf{d}_{\rm E}^T]^T$, and

$$C_{\rm E} = \begin{bmatrix} 1 & (\mathbf{r}_1 - \mathbf{r}_{\rm E})^T \\ 1 & (\mathbf{r}_3 - \mathbf{r}_{\rm E})^T \\ 1 & (\mathbf{r}_4 - \mathbf{r}_{\rm E})^T \end{bmatrix}. \tag{37}$$

Let $D_{\rm E}$ be the 4×3 matrix with its three row vectors given by $\frac{1}{2}((\mathbf{r}_i - \mathbf{r}_{\rm E}) \otimes (\mathbf{r}_i - \mathbf{r}_{\rm E}))^T$ for $i = 1,3,4$. Then $\widehat{\mathbf{z}}_{\rm E} = C_E \mathbf{s}_{\rm E} + D_E \vec{H}^{\rm P}$ which implies that $\mathbf{s}_{\rm E} = C_{\rm E}^{-1} [\widehat{\mathbf{z}}_{\rm E} - D_E \vec{H}^{\rm P}]$.
3) Along the positive or negative direction of $\mathbf{d}_{\rm E}$, we may find the point $\mathbf{r}_{\rm J}$ (Fig. 1) where $z_{\rm J} = z_{\rm c}^{\rm P}$ using

$$\mathbf{r}_{\rm J} = \mathbf{r}_{\rm E} + (z_{\rm c}^{\rm P} - z_{\rm E}) \frac{\mathbf{d}_{\rm E}}{\| \mathbf{d}_{\rm E} \|}. \tag{38}$$

4) Estimate $\mathbf{d}_{{\rm J},k}$ by solving the following equations for $i = 1,3,4$:

$$\widehat{z}_i = z_{\rm J} + \mathbf{d}_{\rm J} \cdot (\mathbf{r}_i - \mathbf{r}_{\rm J}) + \frac{1}{2} (\mathbf{r}_i - \mathbf{r}_{\rm J})^T H^{\rm P} (\mathbf{r}_i - \mathbf{r}_{\rm J}). \tag{39}$$

5) Repeat the steps 2), 3) and 4) for the formation consisting $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ with appropriate changes in the subscripts for points $\mathbf{r}_\mathrm{F}$ and $\mathbf{r}_\mathrm{K}$ (Fig. 1).
6) Let $\mathbf{y}_{1\mathrm{J}}$, $\mathbf{y}_{1\mathrm{K}}$ and $\mathbf{y}_{1\mathrm{c}}$ denote the unit vectors along the directions of the gradient $\mathbf{d}_\mathrm{J}$, $\mathbf{d}_\mathrm{K}$ and $\mathbf{d}_\mathrm{c}$. Define $\delta\theta_\mathrm{L} = \arccos(\mathbf{y}_{1\mathrm{J}} \cdot \mathbf{y}_{1\mathrm{c}})$, $\delta s_\mathrm{L} = \|\mathbf{r}_\mathrm{J} - \mathbf{r}_\mathrm{c}\|$, $\delta\theta_\mathrm{R} = \arccos(\mathbf{y}_{1\mathrm{K}} \cdot \mathbf{y}_{1\mathrm{c}})$, and $\delta s_\mathrm{R} = \|\mathbf{r}_\mathrm{K} - \mathbf{r}_\mathrm{c}\|$. Obtain the estimate for $\kappa(s)$ at $\mathbf{r}_\mathrm{c}$ as
$$\kappa_\mathrm{c} = \frac{1}{2}\left(\frac{\delta\theta_\mathrm{L}}{\delta s_\mathrm{L}} + \frac{\delta\theta_\mathrm{R}}{\delta s_\mathrm{R}}\right). \tag{40}$$
Obtain the estimate for H_xx according to (27).
7) Approximate $\frac{d}{ds}\|\nabla z_\mathrm{c}\|$ by
$$\frac{d}{ds}\|\nabla z_\mathrm{c}\| = \frac{\|\mathbf{d}_\mathrm{K}\| - \|\mathbf{d}_\mathrm{J}\|}{\delta s_\mathrm{L} + \delta s_\mathrm{R}}. \tag{41}$$
Then using (29), the estimate H_xy is
$$H_\mathrm{xy} = \frac{\|\mathbf{d}_\mathrm{K}\| - \|\mathbf{d}_\mathrm{J}\|}{\delta s_\mathrm{L} + \delta s_\mathrm{R}}. \tag{42}$$
8) Solve
$$\widehat{z}_i = z_\mathrm{c} + \mathbf{d}_\mathrm{c} \cdot (\mathbf{r}_i - \mathbf{r}_\mathrm{c}) + \frac{1}{2}(\mathbf{r}_i - \mathbf{r}_\mathrm{c})^T H_\mathrm{c}(\mathbf{r}_i - \mathbf{r}_\mathrm{c}) \tag{43}$$
for H_yy where $i = 1,2,3,4$.

The resulting matrix $H_{\mathrm{c},k}$ can be used directly as the estimate for the Hessian at the kth step. Or we may repeat steps 1)-8) starting from $H_{\mathrm{c},k}$ to get a new estimate for the curvature and then to improve the estimate $H_{\mathrm{c},k}$. The procedure becomes an iterative numerical algorithm that solves the set of nonlinear equations that z_c, $H_{\mathrm{xx},k}$, $H_{\mathrm{xy},k}$ and $H_{\mathrm{yy},k}$ satisfy given $\widehat{z}_{i,k}$ for $i = 1,2,3,4$. The prediction from step $k-1$ provides a reasonable initial value for this iterative algorithm.

V. Formation and Control

The estimation process imposes constraints on feasible platform formations, and the shape of the formation affects error in the estimates. We may design special formations to reduce complexity in theoretical analysis, computation, and operation.

A. The Cross Formation

As an example of such special formations, we arrange the four platforms in a symmetric formation as shown in Figure 2 and choose a coordinate frame attached to the formation so that $\|\mathbf{r}_{2,k} - \mathbf{r}_{\mathrm{c},k}\| = \|\mathbf{r}_{\mathrm{c},k} - \mathbf{r}_{1,k}\| = a$ and $\|\mathbf{r}_{3,k} - \mathbf{r}_{\mathrm{c},k}\| = \|\mathbf{r}_{\mathrm{c},k} - \mathbf{r}_{4,k}\| = b$. Then, in the Frenet-Serret coordinate frame, C_k and D_k have very simple form because of the symmetry. However, we do need control laws to stabilize this special formation.

B. Formation Control

We view the entire formation as a deformable body. The shape and orientation of this deformable body can be described using a special set of Jacobi vectors, c.f. [17]–[21] and the references therein. Here, *assuming that all platforms have unit mass*, we define the set of Jacobi vectors as $\mathbf{q}_1 = \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_1)$, $\mathbf{q}_2 = \frac{1}{\sqrt{2}}(\mathbf{r}_3 - \mathbf{r}_4)$, and $\mathbf{q}_3 = \frac{1}{2}(\mathbf{r}_4 + \mathbf{r}_3 - \mathbf{r}_2 - \mathbf{r}_1)$.

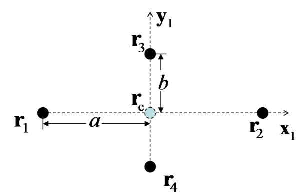

Fig. 2. *A symmetric arrangement of the formation to simplify the equations for estimates.*

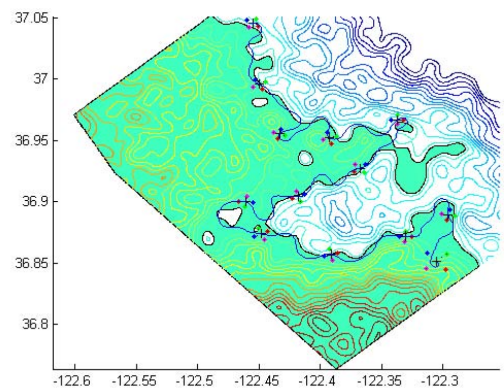

Fig. 3. *Tracking the temperature level curve of 14.25°C in an estimated temperature field near Monterey Bay, CA on August 13, 2003. For visualization purpose, the level curve is accentuated. The trajectory of the center of formation is plotted with status of the formation shown along the trajectory. The horizontal axis corresponds to longitude and the vertical axis to latitude.*

Lagrange's equations for the formation in the lab frame are simply the set of Newton's equations: $\ddot{\mathbf{r}}_i = \mathbf{f}_i$ where $\mathbf{f}_i$ is the control force for the ith platform for $i = 1,2,3,4$. In terms of the Jacobi vectors, these equations are equivalent to
$$\begin{aligned}\ddot{\mathbf{q}}_j &= \mathbf{u}_j \\ \ddot{\mathbf{r}}_\mathrm{c} &= \mathbf{f}_\mathrm{c}\end{aligned} \tag{44}$$
where $j = 1,2,3$ and $\mathbf{u}_j$ and $\mathbf{f}_\mathrm{c}$ are equivalent forces satisfying $\mathbf{f}_1 = \mathbf{f}_\mathrm{c} - \frac{1}{\sqrt{2}}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_3$, $\mathbf{f}_2 = \mathbf{f}_\mathrm{c} + \frac{1}{\sqrt{2}}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_3$, $\mathbf{f}_3 = \mathbf{f}_\mathrm{c} + \frac{1}{\sqrt{2}}\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3$, and $\mathbf{f}_4 = \mathbf{f}_\mathrm{c} - \frac{1}{\sqrt{2}}\mathbf{u}_2 + \frac{1}{2}\mathbf{u}_3$.

We now design the control forces $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ so that as $t \to \infty$, $\mathbf{q}_1(t) \to \frac{a^*}{\sqrt{2}}\mathbf{x}_1$, $\mathbf{q}_2(t) \to -\frac{b^*}{\sqrt{2}}\mathbf{y}_1$, and $\mathbf{q}_3(t) \to \mathbf{0}$ where $\mathbf{x}_1$ and $\mathbf{y}_1$ are tangent and normal vectors for the level curve at point $\mathbf{r}_\mathrm{c}$, and a^* and b^* are desired values for a and b. Assuming that $\mathbf{x}_1$ and $\mathbf{y}_1$ are slowly varying, a simple controller is $\mathbf{u}_1 = -k_2(\mathbf{q}_1 - \frac{a^*}{\sqrt{2}}\mathbf{x}_1) - k_3\dot{\mathbf{q}}_1$, $\mathbf{u}_2 = -k_2(\mathbf{q}_2 + \frac{b^*}{\sqrt{2}}\mathbf{y}_1) - k_3\dot{\mathbf{q}}_2$, and $\mathbf{u}_3 = -k_2\mathbf{q}_3 - k_4\dot{\mathbf{q}}_3$, where k_2, k_3 and k_4 are positive, constant, scalar gains. We have proved that this controller achieves the formation asymptotically with an exponential rate of convergence by following an approach suggested in [22]. We design $\mathbf{f}_\mathrm{c}$ to implement the speed and steering control as in Section II so that the center of the formation tracks a level curve.

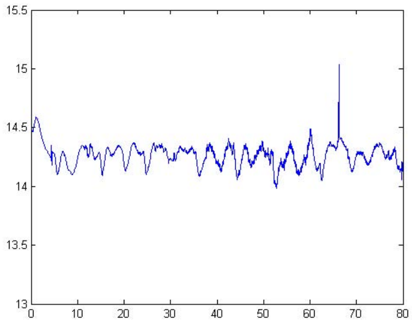

Fig. 4. *The estimate z_c (°C) versus time (hour).*

VI. Simulation Results

The level curve tracking algorithm is applicable to adaptive sampling using sensor networks in the ocean. Adaptive ocean sampling is a central goal of our collaborative Adaptive Sampling and Prediction (ASAP) project [23]. The latest ASAP field experiment took place in August 2006 in Monterey Bay, California. Ten gliders and other propelled underwater vehicles were employed to carry on a series of scientific experiments for oceanographic research for one month. A level curve tracking mission may be carried out in future ASAP experiments.

In order to test our current algorithms on realistic ocean fields, we use a snapshot of the temperature field near Monterey Bay produced by the Harvard Ocean Prediction System (HOPS) [24]. This field reflects the temperature at 20 meters below sea surface on 00:00 August 13th, 2003. It has incorporated glider measurements during the Autonomous Ocean Sampling Network (AOSN) field experiment [1]. We have added spatially correlated noise to the HOPS field.

Four platforms are employed to track a level curve with temperature 14.25 °C. The trajectory of the formation center and the shape of the formation are plotted in Figure 3. We control the center of the formation to travel at 1 km per hour. Figure 4 shows the estimates of the temperature at the center of the formation versus time. One can see the estimates centered around 14.25°C with small error. In the real environment, the ocean field will be time varying. The rate of the changing ocean field is usually slower as compared to the speed of the platforms. Although we have used a static field in our simulation, the algorithm will be applicable to a slowly varying ocean field.

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